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Research Article

Fuzzy Primal and Fuzzy Strongly Primal Ideals

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Abstract: Primal ideal has been introduced and discussed in the literature. In the present note we commence first by giving a concept of strongly primal ideal which is followed by illustrating its implications with strongly prime ideals, strongly primary, and also with strongly irreducible ideals. In addition, we introduce fuzzy primal ideal, fuzzy strongly primal ideal, and fuzzy almost primal ideal and also discuss their relations among each other and with the other existing fuzzy ideals in the literature.

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1. INTRODUCTION AND PRELIMINARIES

Following [1] an element a is called prime to an ideal I, if $ab\in I$ implies $b\in I$ and an ideal II is a primal ideal if the elements which are not prime to I form an ideal P called an adjoint ideal of I. An element $a\in R$ is said to be almost prime to an ideal I provided that $ra\in I\text{-}I^2$ (with $r\in R$) implies that $r\in I$. An ideal I of a ring R is called an almost prime if $xy\in I\text{-}I^2$ implies either $x\in I$ or $y\in$I [2, definition 2.4]. If A (I) denotes the set of all elements of R that are not almost prime to I, I is called an almost primal ideal of R if the set A(I) $\cup$ I^2 forms an ideal of R [3]. A proper ideal I of R is said to be almost primary ideal if whenever ab$\in I$-I^2, then $a\in I$ or $b\in$Rad(I) [4, definition 2.1].

An ideal I of a commutative ring R is said to be irreducible if an ideal I is not the intersection of two ideals of R that properly contain it. An ideal I of a ring R is irreducible if, whenever I is the intersection of two ideals i.e., $I = J\cap K$, then either $I = J$or $I = K$. An ideal I of a ring R is said to be a strongly irreducible if for ideals J and K of R, the inclusion $J\cap K\subseteq I$ implies that either $J\subseteq$ I or $K\subseteq I$ [5]. By [1, Theorem 1], every irreducible ideal is a primal. A prime ideal P of R is called strongly prime if $xy\in P$, where $x,y\in K$, then $x\in P$ or $y\in P$ (Alternatively P is a strongly prime if and only if $x^{-1}P\subseteq P$ whenever $x\in K\backslash R$ [6, definition, page2]. Similarly an ideal I is a strongly primary ideal, in the sense that $xy\in P$, x, $y\in K$ implies that either $x\in P$ for some $n\geq 1$or $y\in P$.

Throughout R will represent a commutative ring or an integral domain, we will clarify it regularly in our discussion whenever it is required.

The concept of fuzzy sets and fuzzy relations were introduced by Zadeh [7]. Fuzzy subgroup and its properties were discussed by Rosenfeld [8]. After this, the notion of a fuzzy ideal of aring was introduced by Liu, Malik, Mordeson and Mukherjee. Fuzzy relations on rings have been

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introduced by Malik and Mordeson [9]. Fuzzy ideal ξ of a ring R is said to be fuzzy prime, if it is non-constant and for any two fuzzy ideals μ and ν of R, the condition $\mu o \nu \subseteq \xi$ implies that $\mu \subseteq \xi$ or $\nu \subseteq \xi$. It is well known that ξ is fuzzy prime if and only if $\xi(0) = 1$, $\xi_*$is a prime ideal of R and $|\mathrm{Im}(\xi)| = 2$ [10, Theorem 3. 5.5]. Fuzzy ideal ξ is called fuzzy primaryif it is non-constant and for any two fuzzy idealsμ, ν of R, $\mu o \nu \subseteq \xi$, implies $\mu \subseteq \xi$ or $\nu \subseteq \sqrt{\xi}$ [10, Theorem 3.5.5]. Let β be an integral fractionary fuzzy ideal of R then β is strongly primary fuzzy ideal of R if for any fractionary idealsμ, ν of R, $\mu o \nu \subseteq \beta$ implies that $\mu \subseteq \beta$ or $\nu \subseteq \sqrt{\beta}$ [11, Definition 4.1]. A fuzzy ideal μ in a Noetherian ring R is called irreducible if $\mu \neq R$ and whenever $\mu_1 \wedge \mu_2 = \mu$ where μ_1 and μ_2 are fuzzy ideals of R, then $\mu_1 = \mu$ or $\mu_2 = \mu$[12, Definition 4.1]. A proper fuzzy ideal μ of a ring R is said to be strongly irreducible if for each pair of fuzzy ideals σ and θ of R, if $\sigma \wedge \theta \subseteq \mu$ then either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$ [13, definition 2]. We recall few terminologies from [14, 11].

Let $\mu_t = \{x \in R: \mu_t(x) \geq 0\}$, a level set, for every $t \in [0, 1]$. For a subset W of R, let $\chi_W^{(t)}$ be the fuzzy subset of K such that $\chi_W^{(t)}(x) = 1$ if $x \in W$ and $\chi_W^{(t)}(x) = t$ if $x \in K \backslash W$, where $t \in [0, 1)$. Let R be an integral domain, a fuzzy R-submodule β of K (quotient field of R) called a fractionary fuzzy ideal of R if there exists $d \in R$, $d \neq 0$, such that $d_1 o \beta \subseteq \chi_R^{(t)}$ for some $t \in [0,1)$. If β be a fractionary fuzzy ideal of R then $\beta|_R$ is a fuzzy ideal of R. If $\beta|_R$ is a prime (maximal) fuzzy ideal of R, then β is called a prime (maximal) fractionary fuzzy ideal of R. If $\beta(x) = 0$ for all $x \in K \backslash R$, then β is called an integral fractionary fuzzy ideal of R. Thus, if β s a prime (maximal) integral fractionary fuzzy ideal of R, then $\mathrm{Im}(\beta) = \{0, 1, t\}$ for some $t \in [0, 1)$.

In section 2, we introduce and discuss strongly primal ideal and its relations with strongly prime ideals, strongly primary, and also with strongly irreducible ideals. In section 3, we introduce fuzzy primal ideal, fuzzy strongly primal, and fuzzy almost primal ideals. All basic notations, terminologies and definitions of fuzzy are referred to [7, 9, 12, 14].

2. STRONGLY PRIMAL IDEAL

In this section we introduce strongly primal ideal and describe its few characteristics. We discuss few relations of strongly primal ideal with strongly prime, strongly primary, and Strongly irreducible ideals.

Definition 1. An element $x \in K \backslash R$ is said to be a strongly prime to an ideal I of a ring R if $xy \in I$ then $yI \subset I$, for all $x, y \in K \backslash R$.

Remark 1. Every strongly prime element to an ideal I is a prime to I.

In case of integral domain we may define strongly prime element as:

Definition 2. We call an element x is a strongly prime to an ideal I of an integral domain R if,

$xy \in I$ implies that $y \in I$, for all $x, y \in K$(where K is a quotient field of R).

We define a strongly primal ideal as:

Definition 3. An ideal I of a ring (or an integral domain) R is called strongly primal if the elements that are not strongly prime to I form an ideal P called the strongly adjoint ideal of I.

We interpret one important result regarding strongly primal ideal in the lemma below.

Lemma 1. If an ideal I is a strongly primal ideal of a ring R, P is a strongly adjoint to I if and only if $ab \in I$ and $bR \not\subset I \Rightarrow aR \subset P$. Conversely, whenever $a \in P$, there always exist an element $b \in K \backslash R$ not contained in I such that $ab \in I$.

Proof. Straightforward.

Remark 2. If I is a strongly primal ideal of an integral domain R, then bc and $b$are not at the same time prime to I whatever the element c, b- c shall be not strongly prime to I.

Remark 3. Every strongly primal ideal is a primal ideal.

Remark 4. In an integral domain R an ideal I is a strongly primal ideal and P be strongly adjoint to I iff $ab \in I$, $b \notin I$ implies $a \in I$ where a, $b \in K$(quotient field of R) and conversely, whenever $a \in P$, there always exists an element b not in I such that $ab \in I$.

Remark 5. Elements which are not strongly primal to I can be represented into a residue class ring *R/I* the zero factors, so we can define the strongly primal ideal as: In *R/I* the zero factors form an ideal *P/I*, where *P* is a strongly prime ideal.

Proposition 1. If *R* is an integral domain then strongly primary ideal is a strongly primal.

Proof. Indeed, the elements that are not strongly prime to a strongly primary ideal *I* are only the elements of its prime radical (because strongly prime implies prime) which constitute strongly adjoint ideal *P* for a strongly primal ideal *I*.

Hence from the above proposition we conclude that:

Strongly primary⇒strongly primal.

Converse of the above proposition does not hold. We illustrate it with the example.

Example 1. Consider a polynomial ring $R = Q[x, y]$ and consider ideal $I = (x^2, xy)$ is a quasi-primary ideal with the radical (x) and is strongly primal with strongly adjoint ideal (x, y).

Now $xy \in I$ but neither $x \in I$ nor $y^n \in I$ for any n.

When the converse of proposition1 holds? We present a proposition here to clarify the matter.

Proposition 2. A strongly primal ideal is a strongly primary ideal if it is a quasi-prime and

also its prime radical and strongly adjoint (which is a strongly prime and hence a prime) prime ideal coincides.

Proof. Let us assume that *I* is a strongly primal ideal and also a quasi-prime, further it's prime radical is coincides with strongly adjoint prime ideal *P*. We show that I is a strongly primary ideal. Consider $ab \in$I, $b \notin I \Rightarrow$) $a \in P$ (strongly adjoint ideal) $\Rightarrow a \in$rad (I) (by assumption) therefore it follows that $a^n \in I$ for some n. Hence an ideal *I* is a strongly primary.

Proposition 3. Every strongly irreducible ideal in a commutative ring *R* (resp. in an integral domain) is a strongly primal ideal.

Proof. Assume that an ideal *I* is strongly irreducible ideal, also suppose that elements $a,b \in I$ so that *a* and *b* are not strongly prime to *I.* We have to show that an ideal *I* is a strongly primal ideal, since $a\mathrm{R} \subset I$ and $b\mathrm{R} \subset I$. Clearly I: (a) and I: (b) are proper divisors of *I* hence their intersection $I : (a) \cap I : (b) = I : ((a) + (b))$ cannot equal to I it implies that a-b is not strongly prime(⇒ prime) to *I*. Hence the result follows by lemma1 and Remark 2. Similarly we can prove it for any integral domain *R*.

Remark 6. If I is a prime ideal then it is strongly irreducible ⇒every strongly adjoint ideal is a strongly prime and hence prime ⇒ strongly irreducible.

We may express implications as:

$$\begin{array}{ccc} \text{Strongly primary ideal} & \Rightarrow & \text{strongly primal ideal} \\ \Downarrow & & \Uparrow \\ \text{Strongly prime ideal} & \Rightarrow & \text{strongly irreducible ideal} \end{array}$$

3. FUZZY PRIMAL AND FUZZY STRONGLY PRIMAL IDEALS

This section consists of three subsections. In first subsection we introduce fuzzy primal ideal while in second we discuss fuzzy strongly primal. In third section we introduce fuzzy almost primal ideal.

3.1. Fuzzy Primal Ideal

In this subsection first we introduce the concept of fuzzy primal ideal and then we defined some relations of fuzzy primal ideal with fuzzy prime, fuzzy primary and fuzzy irreducible ideals.

We initiate with the following definition.

Definition 4. Let *R* be a fuzzy ring, *I* be a fuzzy ideal of *R*. An element μ is said to be a fuzzy

prime to an ideal *I* if $\nu \wedge \mu \in$I, implies$\nu \in I$.

Definition 5. An ideal *I* is said to be a fuzzy primal ideal of a fuzzy commutative ring *R* if the elements which are not prime to *I* form a fuzzy ideal ζ called an adjoint fuzzy ideal to *I*.

Remark 7. If *I* is a fuzzy primal ideal of a fuzzy ring *R* then $\nu \wedge \mu$ together, with μ is not a fuzzy prime to *I* whatever the element ν , $\nu \wedge \mu$ shall be a non-fuzzy prime to *I*.

Here we define necessary and sufficient conditions for any fuzzy ideal to be a fuzzy primal ideal.

Remark 8. A fuzzy ideal ξ is a fuzzy primal ideal and ζ is a fuzzy adjoint ideal to ξ if and only if for all ν, $\mu\in R$ such that $\mu\Lambda\nu\in I$, $\mu\in\xi$ implies $\nu\notin I$ and conversely, whenever there

exist fuzzy ideal ζ contains μ there always exist an element v such that $\nu\notin I$ and $\mu\wedge\nu\in I$.

Remark 9. It is straightforward that, for any fuzzy ideals μ and ν of R, $\nu\Lambda\mu$together with μ is not a fuzzy prime element to a fuzzy ideal I whatever elementν, their difference μ-ν shall be a non-fuzzy prime to I.

Proposition 4. Every fuzzy adjoint ideal to a fuzzy primal ideal is a fuzzy prime ideal.

Proof. Let R be a fuzzy commutative ring and ζ be a fuzzy adjoint ideal to a fuzzy primal idealξ. Following definition5, $\nu\wedge\mu$implies that $\nu\notin\xi$ and all elements which are not a fuzzy prime to fuzzy ideal I forms an ideal (fuzzy adjoint ideal) ζ such that $\mu\in\zeta\Rightarrow$ for all elements of a fuzzy ring R ;ϑ,θ, η… which are fuzzy prime to a fuzzy ideal I are not contained in ζ, so we have $\theta\wedge\mu\in\zeta\Rightarrow\mu\in\zeta$ and $\theta\notin\zeta$ hence ζ is a prime fuzzy ideal.

Proposition 5. Fuzzy primary ideal of a ring R is a fuzzy primal ideal.

Proof. Suppose ξ is a fuzzy primary ideal clearly ξ is a non-constant and for any two fuzzy ideals ν and μ of R, $\nu\wedge\mu\in\xi$ implies $\mu\in\xi$or$\nu\in\sqrt{\xi}$. Suppose $\mu\in\xi$so $\nu\subseteq\sqrt{\xi} = P$ be a fuzzy adjoint ideal to ξ such that for all elements ;ϑ,θ, η… in a fuzzy ring R, $\vartheta\Lambda\nu\in P$ and thus ξ is a fuzzy primal ideal.

Proposition 6. Every fuzzy irreducible ideal in a fuzzy ring R is a fuzzy primal ideal.

Proof. Suppose I is a fuzzy irreducible ideal of a ring R, also elements μ_1, μ_2 are contained in I so that for all elements ϑ, θ, η… in a ring R we have, $\theta\wedge\mu_1\in I$, and $\theta\wedge\mu_2\in I$. We have to show that I is a fuzzy primal ideal, let $< 1 >= R$, where $< 1 >$ is a fuzzy (improper) principal ideal of R. Since $\mu_1 \wedge< 1 >\in I$ and $\mu_2 \wedge< 1 >\in I$. Clearly I: $\mu_1 \wedge< 1 >$ and I: $\mu_1 \wedge< 1 >$ are proper divisors of I hence their intersection I: $\mu_1\wedge< 1 >\cap I$: $\mu_1 \wedge< 1 > = I$:$< \mu_1 + \mu_2 >\neq I\Rightarrow\mu_1 \wedge\mu_1$ is not a prime to I and thus by remark7 and remark8, I is a fuzzy primal ideal.

3.2. Fuzzy Strongly Primal Ideal

In this section we introduce and discuss fuzzy strongly primal ideal of an integral domain.

We also establish few connections of fuzzy strongly primal ideals.

Definition 6. Let R be a fuzzy ring with quotient field K and I be a fractionary fuzzy ideal of R. An element $\mu\in K$ is said to be a fuzzy strongly prime to an ideal I if $\nu\wedge\mu\in I$, implies $\nu\in I$

(whereν, $\mu\in K$).

Definition 7. A fractionary fuzzy ideal ξ is said to be a fuzzy strongly primal ideal of an integral domain R, if all the elements fuzzy non-prime to an ideal ξ form an ideal ζ (a fuzzy strongly adjoint ideal to a fuzzy strongly primal ideal).

In remark 10, we give necessary and sufficient condition for any fuzzy ideal to become a fuzzy strongly primal ideal.

Remark 10. A fractionaryfuzzy ideal ξ of a fuzzy integral domain R is a fuzzy strongly primal ideal and ζ is a fuzzy strongly adjoint ideal to I if and only if there existμ, $\nu\in K$

(Quotient field of R) such that $\mu\Lambda\nu\in\xi$ and $\mu\notin\xi$ it implies $\nu\in\zeta$ and conversely, whenever $\nu\in\zeta$, there always exist μ not contained in ξsuch that$\mu\Lambda\nu\in\xi$.

Remark 11. If an ideal I is a fuzzy strongly primal ideal then together$\mu\wedge\nu$, withν is not fuzzy strongly prime to I whatever the elementν, $\mu\wedge\nu$shall be not strongly prime to I.

Proposition 7. If R is an integral domain then every fuzzy strongly adjoint ideal to a fuzzy strongly primal ideal is a strongly prime fuzzy ideal.

Proof. Let R be a fuzzy integral domain and ζbe a fuzzy strongly adjoint ideal to fuzzy strongly primal idealξ. Following definition12, we have elementsμ, $\nu\in$K (quotient field of R) such that $\mu\Lambda\nu\in\xi$, $\mu\notin\xi$and there exist a fuzzy strongly adjoint idealζwhich contains ν.

By assumption for all elements (μ,θ, $\eta\ldots\in K$) strongly prime to ξ, $\theta\Lambda\nu\in\zeta$such that$\nu\in\zeta$

$\Rightarrow\zeta$is a strongly prime ideal.

Proposition 8 . In an integral domain R a strongly fuzzy primary ideal is a fuzzy strongly primal ideal.

Proof. Suppose ξ is a strongly fuzzy primary ideal, clearly ξ is non-constant and for any two elementsμ, $\nu\in K$ (quotient field of R),$\mu\wedge\nu\in\xi$ implies $\mu\in\xi$ or$\nu\in\sqrt{\xi}$. Suppose $\mu\notin\xi$ so $\nu\in\sqrt{\xi} =P$ be a fuzzy strongly adjoint ideal toξ, such that for allμ,θ, $\eta\ldots\in K$ fuzzy strongly prime elements toξ, $\theta\wedge\nu\in P=\sqrt{\xi}$ and thus ξ is a fuzzy strongly primal ideal.

Proposition 9. Every fuzzy strongly irreducible ideal in a fuzzy ring R is a fuzzy strongly primal ideal.

Proof. Straightforward as in proposition 8.

3.3. Fuzzy almost Primal Ideal

In this section we introduce fuzzy almost prime ideal of a fuzzy integral domain.

We call an element μ of a fuzzy ring R a fuzzy almost prime to an ideal I if, whenever $\mu\wedge\ \nu\in I\text{-}I^2$ $\Rightarrow\nu\in I$.

We may define few more terminologies.

Definition 8. A fuzzy ideal I of a fuzzy ring R is said to be a fuzzy almost prime if $\mu\wedge\nu\in$ I-I^2 implies either $\mu\in$ I or $\nu\in I$.

Definition 9. A fuzzy ideal ξ is said to be a fuzzy almost primal ideal of fuzzy commutative ring R, if it is non-constant and for any two elementsμ, $\nu\in R$, such that $\mu\wedge\nu\in\xi\text{-}\xi^2$ and whenever $\mu\notin\xi$there exist a fuzzy idealζ such that$\nu\in\zeta$. We call ζ a fuzzy almost adjoint ideal to a fuzzy primal idealξ.

Definition 10. A fuzzy ideal I of a fuzzy ring R is said to be a fuzzy almost primary if whenever $\mu\wedge\nu\in I\text{-}I^2$ implies $\mu\in I$ or $\nu\in$rad (I), whereμ, $\nu\in R$.

Definition 11. Let I be a proper fractionary fuzzy ideal of fuzzy integral domain R, I is said to be a fuzzy strongly almost primary ideal if for anyμ, $\nu\in K$ (quotient field of R) $\mu\wedge\nu\in I\text{-}I^2$ implies$\mu\in$ I or $\nu\in\sqrt{I}$. If $\sqrt{I}=P$ then we call I a strongly almost P-primary ideal.

Here we discuss few relations among the defined ideals.

Proposition 10. Fuzzy almost primary ideal of a fuzzy ring R is a fuzzy almost primal ideal.

Proof. Let ξ be a fuzzy almost primary ideal of fuzzy ring R. Clearly $\mu\wedge\nu\in\xi\ \text{-}\xi^2$ implies $\mu\in\xi$or$\nu\in\sqrt{\xi}$. Assume $\mu\notin\xi$ it implies $\nu\in\sqrt{\xi}=P$ be a fuzzy almost adjoint ideal to ξ such that for all elements ξ (μ, θ, $\eta\ldots$) contained in R, $\theta\wedge\nu\in\sqrt{\xi}=P$, this implies that ξ is a fuzzy almost primal ideal.

Proposition 11. Every fuzzy almost adjoint ideal to a fuzzy almost primal ideal is a fuzzy almost prime ideal.

Proof. Let R be a fuzzy ring and ζ be a fuzzy almost adjoint ideal to fuzzy almost primal ideal ξ. Following definition11, we have $\mu\wedge\nu\in\xi\text{-}\xi^2$ implies that there exist a fuzzy adjoint ideal ζsuch that$\mu\notin\xi$ there exist a fuzzy idealζ such that$\nu\in\zeta$ this implies that for all μ, θ, $\eta\ldots$ not contained in ζ, we have $\theta\wedge\nu\in\zeta$ implies $\nu\in\zeta\Rightarrow\zeta$ is a fuzzy prime ideal.

4. REFERENCES

1. Fuchs, L. On primal ideals. *Proceedings of American Mathematical Society* 1: 1-6 (1950).
2. Bhatwadekar, S.M. & K. Sharma. Unique factorization and birth of almost primes. *Communications in Algebra* 33: 43-49 (2005).
3. Darani, A.Y. Almost Primal Ideals in Commutative Rings. *Chiang Mai Journal of Science* 38(2): 161-165 (2011).
4. Bataineh. M. & Safa Kuhail. Generalizations of Primary Ideals and Sub-modules. *International Journal of Contemporary Mathematical Sciences* 6(17): 811-824 (2011).
5. Heinzer, W.J., L.J. Ratli Jr., & D.E. Rush. Strongly irreducible ideals of commutative Rings. *Journal of Pure and Applied Algebra* 166: 267-275(2002).
6. Hedstrom, J.R. & E.G. Houston. Pseudo-valuation domains. *Pacific Journal of Mathematics* 75 (4): 551-567 (1978).
7. Zadeh, L.A. Fuzzy sets. *Information and Control* 8: 383-353 (1965).
8. Rosenfeld, A. Fuzzy groups. *Journal of Mathematical Analysis and Applications* 35: 512-517 (1971).
9. Malik, D.S. & J.N. Mordeson. Fuzzy relations on rings and groups. *Fuzzy sets and Systems* 43: 117-123 (1991).
10. Mordeson, J. N. & D.S. Malik. *Fuzzy Commutative Algebra*. World Scientific Publishing,Singapore (1998).
11. Kim, M.O. & H. Kim. Strongly prime fuzzy ideals

and related fuzzy ideals in an integral domain. *Journal of Chungcheong Mathematical Society* 22(3): 333-351(2009).
12. Muraly, V. & B.B. Makaba. On Krull intersection theorem of fuzzy ideals. *International Journal of Mathematics and Mathematical Sciences* 4: 251-262 (2003).
13. Shah, T. & M. Saeed. Fuzzy ideals in Laskerian rings, *Mathematiqki Vesnik* 65(1): 74–81 (2013).
14. Lee, K.H. & J.N. Mordeson, J. N. Fractionary fuzzy ideals and Dedekind domains. *Fuzzy Sets and Systems* 99: 105-110 (1998).