Proceedings of the Pakistan Academy of Sciences 50 (4): 339–344 (2013)

ISSN: 0377 - 2969 (print), 2306 - 1448 (online)

Pakistan Academy of Sciences

Research Article

Borel Summability for Fractional Differential Equation in the Unit Disk

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Abstract: In this article, we consider some classes of nonlinear fractional differential equations with singularity take the form

$$t^{\alpha} \frac{\partial^{\alpha} u(t,z)}{\partial t^{\alpha}} = F(t,z,u,\frac{\partial u}{\partial z}), \quad 0<\alpha<1,$$

where $t \in J := [0,1]$ and $z \in U := \{z \in \mathbb{C} : |z| < 1\}$. Our purpose is to establish a result similar to the k-summability known in the case of singular ordinary differential equations. It's shown that, under some conditions, all formal solutions are Borel summable or k-summable with respect to $z \in U$ in all directions except at most a countable number.

Keywords: Fractional calculus; Fractional differential equation; Holomorphic solution; Unit disk; Riemann-Liouville operators; Nonlinear; Singular fractional differential equation; Borel summable.

AMS Mathematics Subject Classification: 30C25

1. INTRODUCTION

Since the last decade, fractional calculus is a rapidly growing subject of interest for physicists and mathematicians. The reason for this is that problems may be discussed in a much more stringent and elegant way than using traditional methods. Fractional differential equations have emerged as a new branch of applied mathematics which has been used for many mathematical models in science and engineering. In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations. The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved.

The present paper deals with a nonlinear singular fractional differential equation [1,2], in sense of the Riemann-Liouville operators, in the analytic category. The Riemann-Liouville fractional derivative could hardly pose the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann-Liouville operators [3].

Definition 1.1. The fractional (arbitrary) order integral of the function f of order $\alpha > 0$ is defined by

Received, January 2013; Accepted, October 2013

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$$I_a^\alpha f(t)=\int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau.$$

When $a=0$, we write $I_a^\alpha f(t)=f(t)*\phi_\alpha(t)$, where $(*)$ denoted the convolution product,

$$\phi_\alpha(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}, t>0$$

and $\phi_\alpha(t)=0, t\le 0$ and $\phi_\alpha \to \delta(t)$ as $\alpha\to 0$ where $\delta(t)$ is the delta function.

Definition 1.2. The fractional (arbitrary) order derivative of the function f of order $0<\alpha<1$ is defined by

$$D_a^\alpha f(t)=\frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau=\frac{d}{dt} I_a^{1-\alpha} f(t).$$

Remark 1.1. From Definition 1.1 and Definition 1.2, we have

$$D^\alpha t^\mu=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \mu>-1; 0<\alpha<1$$

and

$$I^\alpha t^\mu=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \mu>-1; \alpha>0.$$

In the present work we consider the summability of fractional differential equation takes the form

$$t^\alpha \frac{\partial^\alpha u(t,z)}{\partial t^\alpha}=F(t,z,u,\frac{\partial u}{\partial z}), \tag{1}$$

subject to the initial condition $u(0,0)=0$, where $t\in J:=[0,1]$, $z\in U$, $u(t,z)$ is an unknown function and $F(t,z,u,v)$ is a function with respect to the variables $(t,z,u,v)\in J\times U\times \mathbb{C}^2$.

We need the following assumptions in the sequel:

(H1) $F(t,z,u,v)$ is a holomorphic function defined in a neighborhood of the origin $(0,0,0,0)\in J\times U\times \mathbb{C}^2$.

(H2) $F(0,z,0,0)\equiv 0$ near $z=0$.

Thus the function $F(t,z,u,v)$ may be expressed in the form:

$$F(t,z,u,v)=A(z)t+B(z)u+C(z)v+R_2(t,z,u,v), \tag{2}$$

where

$$A(z):=\frac{\partial F}{\partial t}(0,z,0,0),\quad B(z):=\frac{\partial F}{\partial u}(0,z,0,0),$$
$$C(z):=\frac{\partial F}{\partial v}(0,z,0,0),$$

and the degree of $R_2(t,z,u,v)$ with respect to (t,z,u,v) is greater than or equal to 2.

(H3) $C(z):=z^2c(z),\quad c:=c(0)\neq 0.$

(H4) Denotes $b:=B(0)$ such that $b\neq \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)},\quad \forall k\in \mathbb{N}^*.$

2. PRELIMINARIES

In the section, we give out some denotations and preparations such as definitions and lemmas.

Definition 2.1. The majorant relations described as: if $a(x)=\sum a_i x^i$ and $A(x)=\sum A_i x^i$, then we say that $a(x) \ll A(x)$ if and only if $|a_i|\le|A_i|$ for each i.

Definition 2.2. [4] Let $C[[t;z]]$ be the formal power series of t,z and $C\{t;z\}$ be the convergent formal power series in some polydisc with positive radius. For a formal power series

$$u(t,z)=\sum_{m\ge 0,n\ge 0} u_{mn} t^m z^{n+1},\quad (t\in J, z\in U)$$

then the formal Borel transform order k in z is

$$\mathrm{B}_k(u)(t,z)=\sum_{m\ge 0,n\ge 0}\frac{u_{mn}}{\Gamma((1+n)/k)} t^m \zeta^{n+1}.$$

and

$$C\{t\}[[z]]_k:=\{u(t,z)\in C[[t;z]] \quad and$$

$$\mathrm{B}_k(u)(t,z)\in C\{t;z\}\},\quad (t\in J, z\in U).$$

Moreover, we say that $u(t,z)\in C\{t\}_r[[z]]_k$ iff there exists a constant $d>0$ such that $\mathsf{B}_k(u)(t,z)$ is holomorphic in $J\times\{z\in U:|z|\le d<1\}$.

Definition 2.3. For $\delta\in\mathsf{R},\theta>0,$ we define sectorial domain by

$$S(\delta,\theta;1)\equiv S(\delta,\theta):=\{z\in U:|\arg(z)-\delta|<\frac{\theta}{2}\}.$$

Here δ,θ are called the direction and opening angle of the sectorial domain $S(\delta,\theta)$, respectively. Note that the radius is equal to 1 ($|z|<1$). A sectorial domain S' is called a proper subsector of $S(\delta,\theta)$ if its closure is contained in $S(\delta,\theta)\cup\{0\}$.

Definition 2.4. Let $u(t,z)$ be analytic on $J\times S(\delta,\theta)$. Then $u(t,z)\in C\{t\}[[z]]_k$ is called a Gevery asymptotic expansion order k of $u(t,z)$ as $z\to 0\in S(\delta,\theta)$, if for any proper subsector S' of $S(\delta,\theta)$ (with sufficiently small radius), there exist positive constants K_1,K_2 and $0<r\le 1$ such that $u(t,z)\in C\{t\}_r[[z]]_k$ and

$$\sup_{t\in J}|u(t,z)-\sum_{n=0}^{N-1}u_n(t)z^n|\le K_1K_2^N N!^{1/k}|z|^N,$$
$$z\in S',N=1,2,3,\ldots$$

This relation is denoted by

$$u(t,z):_k u(t,z).$$

Note that if $u(t,z):_k u(t,z)$ then $u(t,z)$ is unique and it is called the k-sum of $u(t,z)$ in the direction δ, and $u(t,z)$ is said to be k-summable in the direction δ. Furthermore, 1-summable is called Borel summable.

The k-summability of $u(t,z)\in C\{t\}[[z]]_k$ in a direction can be characterized as follows:

Lemma 2.1. A formal series $u(t,z)\in C\{t\}[[z]]_k$ is k-summable in $S(\delta,\theta)(\theta>\pi/k)$ if and only if $\mathsf{B}_k(u)(t,\zeta)$ is analytic in $J\times S(\delta,\theta-\pi/k)$ for all radius and satisfies a growth condition of exponential type

$$\sup_{t\in J}|\mathsf{B}_k(u)(t,\zeta)|\le K_1|\zeta|^{1-k}e^{K_2|\zeta|^k},$$
$$\zeta\in S'\subset S(\delta,\theta-\pi/k) \tag{3}$$

for some positive constants K_1 and K_2.

Definition 2.5. [5] Let $S=S(\delta,\theta)$ and $\mu>0$. We denote by A_μ the space of holomorphic functions f in S such that there exists $C>0$ satisfying

$$|f(\zeta)(1+|\zeta|^2)e^{-\mu|\zeta|}|\le C,\quad\forall\zeta\in S.$$

For positive constant $M<\infty$, we define the norm $\|\ \|_\mu$ in A_μ by the formula

$$\|f\|_\mu=M\sup_{\zeta\in S}|f(\zeta)(1+|\zeta|^2)e^{-\mu|\zeta|}|.$$

Note that $A_\mu,\|\ \|_\mu$ is a Banach algebra with respect to the convolution product. If $\mu_2>\mu_1,\mathsf{A}_{\mu_1}$ can be considered as a sub-space of A_{μ_2} and for any $f\in\mathsf{A}_{\mu_1}$,

$$\|f\|_{\mu_1}\le\|f\|_{\mu_2}.$$

More properties can be found in the following results:

Lemma 2.2. [5,6] If $f,g\in\mathsf{A}_\mu$, then $f*g\in\mathsf{A}_\mu$ and $\|f*g\|_\mu\le\|f\|_\mu\|g\|_\mu$.

Lemma2.3. Let $\mu_2>\mu_1$ and $f\in\mathsf{A}_{\mu_1},g\in\mathsf{A}_{\mu_2}$, then

$$\|f*g\|_{\mu_2}\le e\frac{\|f\|_{\mu_1}\|g\|_{\mu_2}}{M^2(\mu_2-\mu_1)}.$$

Proof. From definition of convolution product we pose

$$
\begin{aligned}
|f*g(\zeta)| &= |\int_0^{\zeta} f(\tau)g(\zeta-\tau)d\tau| \\
&\leq \| f \|_{\mu_1} \| g \|_{\mu_2} \int_0^{|\zeta|} \frac{e^{\mu_1\tau}}{M(1+\tau^2)} \frac{e^{\mu_2(|\zeta|-\tau)}}{M(1+(|\zeta|-\tau)^2)} d\tau \\
&\leq \frac{\| f \|_{\mu_1} \| g \|_{\mu_2}}{M^2} (1+|\zeta|^2)^{-1} e^{\mu_2|\zeta|} \int_0^{|\zeta|} e^{(\mu_1-\mu_2)\tau} d\tau \\
&\leq \frac{\| f \|_{\mu_1} \| g \|_{\mu_2}}{M^2} (1+|\zeta|^2)^{-1} e^{\mu_2|\zeta|} \frac{1-e^{(\mu_1-\mu_2)|\zeta|}}{\mu_2-\mu_1} \\
&\leq \frac{\| f \|_{\mu_1} \| g \|_{\mu_2}}{M^2(\mu_2-\mu_1)} \frac{e^{\mu_2|\zeta|}}{1+|\zeta|^2} \\
&\leq \frac{\| f \|_{\mu_1} \| g \|_{\mu_2}}{M^2(\mu_2-\mu_1)} e^{\mu_2|\zeta|}.
\end{aligned}
$$

Hence for arbitrary $\mu_2 \leq \frac{1}{|\zeta|}$, $|\zeta|<1$ we obtain the desired result.

3. EXISTENCE OF UNIQUE SOLUTION

We have the following result:

Theorem 3.1. Consider Eq.(1), the conditions (H1)-(H4) are valid, then the unique formal solution $u(t,z)$ of Eq.(1) is a Borel summable in all directions except the direction which passes through any point of the set

$$\left\{\frac{\frac{\Gamma(1)}{\Gamma(1-\alpha)}-b}{c}, \frac{\frac{\Gamma(2)}{\Gamma(2-\alpha)}-b}{c}, \frac{\frac{\Gamma(3)}{\Gamma(3-\alpha)}-b}{c}, \frac{\frac{\Gamma(4)}{\Gamma(4-\alpha)}-b}{c}, \ldots\right\}, \quad 0<\alpha<1.$$

Construction. Equation (1) can be written as

$$t^{\alpha}\frac{\partial^{\alpha}u(t,z)}{\partial t^{\alpha}} = A(z)t + B(z)u + c(z)z^2\frac{\partial u(t,z)}{\partial z} + \sum_{m+n+p\geq 2} b_{m,n,p}(z)t^m u^n (z^2\frac{\partial u}{\partial z})^p. \quad (4)$$

Now, we only need to consider the summability of the solution $u(t,z)$ of Eq.(4), with $u(t,0)=u(0,z)=0$. Let $B_1(u)(t,\zeta):=u(t,\zeta)$ be the Borel transform of $u(t,z)$, then equation (4) is reduced into the following convolution product equation:

$$[t^{\alpha}\frac{\partial^{\alpha}}{\partial t^{\alpha}} - (b+c\zeta)]u = A(\zeta)t + B(\zeta)*u + C(\zeta)*(\zeta u) + F(\zeta). \quad (5)$$

If we set $u(t,\zeta)=\sum_{l=1}^{\infty} u_l(\zeta)t^l$; then u_l satisfies the following equation

$$[\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)} - b - c\zeta]u_l(\zeta) = A(\zeta)*u_l(\zeta) + B(\zeta)*u_l(\zeta) + C(\zeta)*(\zeta u_l(\zeta)) + F_l(\zeta) \quad (6)$$

such that

$$|\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)} - b - c\zeta| \geq \sigma(\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)} + |\zeta|). \quad (7)$$

By Lemma 2.1, to prove the k-summability of a formal solution, we only need to prove an estimate (3) of its Borel transform order k.

By Lemma 2.3 and the Banach fixed point theorem, we have

Lemma 3.1. Consider equation (6) and let $\sigma>0$ as in (7). Let μ_0 be a sufficiently large number such that $B, C \in \mathrm{A}_{\mu_0}$ and let $\mu = \mu_0 + 2e[\mathrm{P}B\mathrm{P}_{\mu_0} + \mathrm{P}C\mathrm{P}_{\mu_0}]\sigma^{-1}M^{-2}$. If $F_l \in \mathrm{A}_{\mu}$, then equation (6) has a unique solution $\widetilde{u_l} \in \mathrm{A}_{\mu}$ and

$$\| (\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)} - b - c\zeta)\widetilde{u_l} \| \leq 2\| F_l \|_{\mu}. \quad (8)$$

Proof. Consider $\psi(\zeta) = (\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)} - b - c\zeta)u_l$ and the operator

$$\mathrm{P}: \psi \mapsto A(\zeta)*\frac{\psi(\zeta)}{\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}-b-c} + B(\zeta)*\frac{\psi(\zeta)}{\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}-b-c} + C(\zeta)*\frac{\zeta\psi(\zeta)}{\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}-b-c} + F_l(\zeta).$$

If functions $\psi,\phi\in \mathsf{A}_\mu$ then

$$\|\ \psi-\phi\ \|_\mu \le \frac{e[\|\ B\ \|_{\mu_0}+\|\ C\ \|_{\mu_0}]\sigma^{-1}}{2eM^2[\|\ B\ \|_{\mu_0}+\|\ C\ \|_{\mu_0}]\sigma^{-1}M^{-2}} \|\ \psi-\phi\ \|_{\mu\le\frac{1}{2}\|\ \psi-\phi\ \|_\mu}$$

by Lemma 2.3 and the facts that

$$\left|\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}-b-c\zeta\right|^{-1}\le\sigma^{-1}$$

and

$$\left|\frac{\zeta}{\left|\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}-b-c\zeta\right|}\right|\le\sigma^{-1}.$$

Hence, by the Banach fixed point theorem, the equation (6) has unique solution.

Furthermore,

$$\left\|\ \frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}-b-c\zeta\tilde{u}_l\ \right\|\le\frac{1}{2}\left\|\ \frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}-b-c\zeta\tilde{u}_l\ \right\|+\|\ F_l\ \|_\mu,$$

which implies the inequality (8).

Proof of Theorem 3.1. Condition (H1) implies that there exists $\mu_0>0$ such that, in (5), all coefficient functions $A,B,C,A_{m,n,p}$ are in A_{μ_0} and that

$$\sum\nolimits_{m+n+p\ge2}\|\ A_{m,n,p}\ \|_{\mu_0}t^m u^n v^p\in C\{t,u,v\}.$$

In view of Lemma 3.1, by indiction we have $(\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}-b-c\zeta)u_l\in\mathsf{A}_\mu$. Now by using the majorant series relationship we have, set

$$U(t):=\sum_{l\ge1}\|\ u_l\ \|_\mu t^l \quad and \quad W(t):= \sum_{l\ge1}\left\|\ \left(\frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}-b-c\right)u_l\ \right\|_\mu t^l.$$

Then by Lemma 2.2 and inequality (8) yields

$$U(t)\ll\sigma^{-1}W(t)\ll2\sigma^{-1}[\|\ A\ \|_{\mu_0}+\sum_{m+n+p\ge2}(|B_{n,m,p}|+\|\ A_{m,n,p}\ \|_{\mu_0})t^mU^n(\sigma^{-1}W)^p].$$

Consequently, we have

$$U(t)\ll X(t):=2\sigma^{-1}[\|\ A\ \|_{\mu_0}+\sum_{m+n+p\ge2}(|B_{n,m,p}|+\|\ A_{m,n,p}\ \|_{\mu_0})t^mX^{n+p}],$$

$$X(0)=0.$$

In virtue of implicit function theorem there is a constant ρ and for all $t\in J$ such that

$$\sup_{t\in J}|X(t)|\le\rho\Rightarrow\sup_{t\in J}|U(t)|\le\rho.$$

Hence

$$\sup_{t\in J}\left|\sum_{l\ge1}\|\ \tilde{u}_l\ \|_\mu t^l\right|\le\rho,$$

which implies the estimate (3) holds and this completes the proof of Theorem 3.1.

Example 3.1. Assume the following equation

$$\begin{cases}\frac{u(t,z)}{1.128}t^{0.5}\frac{\partial^{0.5}u(t,z)}{\partial t^{0.5}}+16z^2\frac{\partial u(t,z)}{\partial z}= \\ zt+(1+z)t^2,\ t\in J=[0,1] \\ \\ u(0,z)=0,\ in\,a\,neighborhood\,of\quad z=0.\end{cases} \tag{9}$$

where $u(t,z)$ is the unknown function. By putting

$$u(t,z)=\mu(z)t+v(t,z)\quad(v(t,z)=O(t^2))$$

as a formal solution. Therefore, $\mu(z)$ satisfies

$$\mu(z)^2+16z^2\mu'(z)-1-z=0.$$

Now by assuming

$$\mu(z):=q+\psi(z),$$

where q is a constant and $\psi(z)=O(z)$ we obtain that $q=\pm1$. Hence we impose the following equations:

$$\begin{cases} 16z^2\psi'(z) + 2\psi(z) = z - \psi^2(z), \; q = 1 \\ \\ \psi(0) = 0, \end{cases} \tag{10}$$

$$\begin{cases} 16z^2\psi'(z) - 2\psi(z) = z - \psi^2(z), \; q = -1 \\ \\ \psi(0) = 0. \end{cases} \tag{11}$$

where the holomorphic solution $\psi(z)$ exists uniquely and converges in a neighborhood of the origin and Borel summable.

4. ACKNOWLEDGEMENTS

The second author was fully supported by LRGS/TD/2011/UKM/ICT/03/02.

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