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Rudiments of N-framed soft sets

Muhammad Saeed
Department of Mathematics
University of Management and Technology, Lahore, Pakistan.
Email: muhammad.saeed@umt.edu.pk

Muhammad Rayees Ahmad
Department of Mathematics
University of Management and Technology, Lahore, Pakistan.
Email: f2018265003@umt.edu.pk

Muhammad Saqlain
Department of Mathematics
Lahore Garrison University, Lahore, Pakistan.
E-mail: msaqlain@lgu.edu.pk

Muhammad Riaz
Department of Mathematics
University of the Punjab Lahore, Pakistan.
Email: mriaz.math@pu.edu.pk

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Abstract. Uncertainty is a human instinct that prevails on the mind of a person while making an important decision. Decision making is the most integral part of human life which has a potential to change an entire life of a man. Soft set theory is an important tool which deals with uncertainty and helps to make the appropriate decision. This paper covers the generalization of soft sets including double framed and triple framed soft sets and discusses its extension to N-framed soft sets. By defining significant aggregative operations including union, intersection, not set, complement, relative complement, difference of double framed, triple framed and in general N-framed soft sets and some other laws, this paper provides an imperative insight for studying N-framed soft sets.

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1. INTRODUCTION

Decisions [4, 9, 25, 28, 21] are central feature of our life and all of us make different decisions in life at different stages. In our childhood, decisions are made by our parents and afterwards, we make our decisions for ourselves for different occasions in life. Successful life is dependent on different good decisions and one solid decision can positively change the course of our life whereas a bad decision can undo all success in a blink. On the other hand, fears are human instinct and we are always surrounded by fears, doubts and vulnerabilities' especially at the time of making an important decision of our life. We also want to experience new things in life and we are always curious to taste different things in life. One cannot ensure the consequences of each decision, but one can minimize the chances of failures.

In order to avoid or minimizes the dangers we prefer to find different means which could help us to overcome issues. In this regard, suggestions, examples of other's decisions help us to make solid choices, but simultaneously different disciplines and theories also guide us in this regard.

Soft set theory [2, 9, 10, 27, 32] is an interesting and emerging theoretical concept which helps us to deal with uncertainties [31] in life and guide us to make productive decisions and choices in life. In recent years, many researchers have studied the fundamentals of soft set theory [2, 9, 10, 27]. It is an efficient tool to deal with the problems of uncertainty and unpredictability, when compared the classical mathematical tools.

In 1999, concept of soft set theory was presented by Molodsov [2, 9, 10, 27] who considered soft set as parameterized family of subset of universe of discourse. Soft sets are helpful in different areas including artificial intelligence, game theory and decision-making problems [9] and it helps to define different functions for different parameters and uses values against defined parameters to manage different issues and decisions in life.

In 2003, Maji et al. [8, 9] pioneer of decision making, introduced operations (union, intersection, complement, relative complement, difference etc.). Properties of these operations and binary information table of soft set were shared to find optimal object, and a decision maker judges the objects of universe under particular conditions. Later, in 2005, Pie and Miao [**?**] advanced the relationship between information system and soft set.

In 2009, Ali at el. [1] introduced novel function for soft set theory including complement, union, intersection, relative complement, difference etc. Later on, Cagman and Enginoglu [2, 3] in 2010 explored soft matrix theory which proved itself a very significant dimension in solving problems while making different decisions. In 2012, Singh and Onyeozili [11, 26] express that the functions of soft set including union, intersection, complement, relative complement, difference etc. is equivalent to the corresponding soft matrices. From Molodsov [2, 9, 10, 27] to present, many practical applications related to soft set theory are applied in various disciplines including sciences and information technology.

Although, many researchers have done a great job in developing numerous concepts of soft set theory, yet there is still a lot of space in this area. In addition, researchers also argue that uncertainties cannot be handled merely with the classical tools of mathematics but may be dealt with the help of existing theories like probability theory, intuitionistic theory, fuzzy set theory [30, 31], soft set theory [2, 9, 10], soft set topology [14, 15, 16, 17, 18, 19, 20, 22] and theory of rough sets [12, 13, 17, 18].

The real world is too complex to understand completely. In this era, the classical theories are not able to compliance with our required targets in various disciplines. In-spite of various advancements, soft set theory can be extended to two or more functions at the same time. In certain situations, the need is to develop two or more functions for the same defined parameters. Some extensions of soft sets into hypersoft sets and thier applications are also proposed by [23, 24]. N-framed soft set is a significant tool that offers more functions simultaneously. Besides, it gives choices to understand the phenomenon and to make an informed decision.

In our life, we encounter so many factors affecting our daily life. For example, a person gives his blood sample for blood test to check its glucose level before eating any food and then he again gives the sample for the same test after consuming some food. Different parameters are defined which remain the same in both tests. However, the results of blood sugar test are different before and after eating indicated by crisp values 0 or 1. Here we need more functions to define a soft set known as double framed soft set which offers two functions on the same set in the form of crisp values. For this purpose, the idea of double framed soft sets has been applied by by Jun and Ahn in BCK/BCI algebras [6].

2. Preliminaries

2.1. **Soft set.** [29, 7, 5]. Suppose $\Im$ a universe of discourse, $\mathcal{T}$ a set of parameters, $P(\Im)$ power set of $\Im$ and $\mathcal{A} \subset \mathcal{T}$ then the pair $(\pi, \mathcal{A})$ where $\pi : \mathcal{A} \to P(\Im)$ is called a soft set over $\Im$ [8, 10].

2.2. **Double framed soft sets (DFSS).** [6] The notion of DFSS was introduced by Jun and Ahn in 2012. In this section, the aggregative operations and matrix representation of double, triple and N-framed soft sets are proposed along with their examples. The definition of DFSS is defined as follows;

A pair $(\ (\ \pi_1, \pi_2\ ), \mathcal{A})$ is called a double framed soft set [6], over a set $\Im$, where π_1 and π_2 are mappings from $\mathcal{A}$ to $P(\Im)$.

2.2.1. *Example 1.* Consider a set $\Im$ contains five cars which are labeled as in the following set $\Im = \{c_1, c_2, c_3, c_4, c_5\}$.
A person is interested to purchase a car from these cars. Given cars possess following characteristics which are mentioned in set, $\mathcal{T} = \{t_1, t_2, t_3, t_4\}$,
where

t_1 represent ‘ beautiful ′
t_2 represent ‘ Cheap ′
t_3 represent ‘ economical ′
t_4 represent ‘ best resale ′

Let $\mathcal{A} \subset \mathcal{T}$, where $\mathcal{A} = \{t_1, t_2, t_4\}$
$\pi_1 : \mathcal{A} \to P(\Im)$, is a mapping given by

$$(\pi_1, \mathcal{A}) = \begin{aligned} &\{\ (\ t_1,\ \{\ c_1, c_2, c_3\ \}\ ),\\ &(\ t_2,\ \{\ c_2, c_3, c_4\ \}\ ),\\ &(\ t_4,\ \{\ c_4, c_5\ \}\ )\ \}, \end{aligned} \tag{2. 1}$$

$\pi_2 : \mathcal{A} \rightarrow P(\Im)$, is a another mapping given by the following relation

$$(\pi_2, \mathcal{A}) = \{ (t_1, \{ c_2, c_3 \}), (t_2, \{ c_2, c_3, c_4 \}), (t_4, \{ c_3, c_4, c_5 \}) \}, \tag{2. 2}$$

then $((\pi_1, \pi_2), \mathcal{A})$ is DFSS.

Remark

(1) If $(\pi_1, \mathcal{A})$ or $(\pi_2, \mathcal{A})$ is a null, then the DFSS becomes a soft set.
(2) (Special Case) If $(\pi_1, \mathcal{A})$ is a soft set and $(\pi_2, \mathcal{A})$ be its negation, then the DFSS becomes a Bi-polar DFSS.

3. Types of DFSS

3.1. **Double framed Soft Subset.** Suppose $((\pi_1, \pi_2), \mathcal{A})$ and $((\Omega_1, \Omega_2), \mathcal{B})$ be DFSS then $((\pi_1, \pi_2), \mathcal{A})$ is double-framed soft subset of $((\Omega_1, \Omega_2), \mathcal{B})$, if

(1) $\mathcal{A} \subset \mathcal{B}$
(2) $\pi_1(t) = \Omega_1(t)$, and $\pi_2(t) = \Omega_2(t) \quad \forall \; t \in \mathcal{A}$

3.1.1. *Example.* Suppose that there are five houses in the initial universe set $\Im$ given by $\Im = \{s_1, s_2, s_3, s_4, s_5,\}$ and let a set of parameters $\mathcal{T} = \{t_0, t_1, t_2, t_3\}$ be a set of status of houses in which

t_0 represents ‘ beautiful ′
t_1 represents ‘ cheap ′
t_2 represents ‘ in good location ′
t_3 represents ‘ in green surroundings ′

Let $\mathcal{A} \subset \mathcal{T}$, where $\mathcal{A} = \{t_0, t_2, t_3\} \subset \mathcal{T}$
where $\pi_1 : \mathcal{A} \rightarrow P(\Im)$ is a mapping such that

$$(\pi_1, \mathcal{A}) = \{ (t_0, \{ s_1, s_2, s_3, s_4, s_5 \}), (t_2, \{ s_1, s_3, s_4 \}), (t_3, \{ s_4, s_5 \}) \}. \tag{3. 3}$$

and $\pi_2 : \mathcal{A} \rightarrow P(\Im)$ is a mapping such that

$$(\pi_2, \mathcal{B}) = \{ (t_0, \{ s_2 \}), (t_2, \{ s_1, s_2, s_3 \}), (t_3, \{ s_1, s_4, s_5 \}) \}. \tag{3. 4}$$

Then $((\pi_1, \pi_2), \mathcal{A})$ is a DFSS. For another DFSS, let $\mathcal{B} = \{t_2, t_3\} \subset \mathcal{T}$.
where $\Omega_1 : \mathcal{B} \rightarrow P(\Im)$ is a mapping such that

$$(\Omega_1, \mathcal{B}) = \{ (t_2, \{ s_1, s_3 \}), (t_3, \{ s_4 \}) \} \tag{3. 5}$$

and $\Omega_2 : \mathcal{B} \rightarrow P(\Im)$ is a mapping such that

$$(\Omega_2, \mathcal{B}) = \{ (t_2, \{ s_1, s_2 \}), (t_3, \{ s_1, s_5 \}) \}. \tag{3. 6}$$

Then $((\Omega_1, \Omega_2), \mathcal{B})$ is a DFSS.
So $((\Omega_1, \Omega_2), \mathcal{B})$ is a double framed soft subset of $((\pi_1, \pi_2), \mathcal{A})$.

3.2. **Double Framed Null soft set.** Let $((\pi_1, \pi_2), \mathcal{A})$ be a DFSS with usual notations, then $((\pi_1, \pi_2), \mathcal{A})$ is called double framed null soft set if and only if,

$$\pi_1(t) = \phi \quad \wedge \quad \pi_2(t) = \phi \quad \forall \quad t \in \mathcal{A}, \tag{3. 7}$$

3.2.1. *Consider statement of example 3.1.1.* Let $\pi_1 : \mathcal{A} \rightarrow P(\Im)$ is a mapping such that

$$(\pi_1, \mathcal{A}) = \{ (t_0, \phi), (t_2, \phi), (t_3, \phi) \}. \tag{3. 8}$$

and $\pi_2 : \mathcal{A} \rightarrow P(\Im)$ is a mapping such that

$$(\pi_2, \mathcal{A}) = \{ (t_0, \phi), (t_2, \phi), (t_3, \phi) \}. \tag{3. 9}$$

4. OPERATION ON DFSS

4.1. **Union of Double Framed Soft Sets.** Let $((\pi_1, \pi_2), \mathcal{A})$ and $((\Omega_1, \Omega_2), \mathcal{B})$ be two DFSS over the same universe $\Im$, then their union is written as $((\pi_1, \pi_2), \mathcal{A}) \cup ((\Omega_1, \Omega_2), \mathcal{B}) = ((\gamma_1, \gamma_2), \mathcal{A} \cup \mathcal{B})$, and is given by;

$$\gamma_i(t) = \begin{cases} \pi_i(t), & \text{if } t \in \mathcal{A} \setminus \mathcal{B} \\ \Omega_i(t), & \text{if } t \in \mathcal{B} \setminus \mathcal{A}, \quad \text{for} \quad \text{i = 1, 2.} \\ \pi_i(t) \cup \Omega_i(t), & \text{if } t \in \mathcal{A} \cap \mathcal{B}. \end{cases}$$

4.1.1. *Example.* Suppose that there are five houses in the initial universe set $\Im$ given by $\Im = \{s_1, s_2, s_3, s_4, s_5, \}$,
Let $\mathcal{T} = \{t_0, t_1, t_2, t_3\}$ be set of parameters, where
t_0 represents ‘ beautiful ′,
t_1 represents ‘ Cheap ′,
t_2 represents ‘ in good location ′,
t_3 represents ‘ in green surroundings ′,
Let us assume that, $\mathcal{A} = \{t_0, t_2, t_3\} \subset \mathcal{T}$
where $\pi_1 : \mathcal{A} \rightarrow P(\Im)$, then the soft set can be written as

$$(\pi_1, \mathcal{A}) = \begin{array}{c} \{ (\ t_0,\ \{\ s_1, s_2, s_3, s_4, s_5\ \}\ ), \\ (\ t_2,\ \{\ s_1, s_3, s_4\ \}\ ), \\ (\ t_3,\ \{\ s_4, s_5\ \}\ )\ \}, \end{array} \tag{4. 10}$$

where $\pi_2 : \mathcal{A} \rightarrow P(U)$, then the soft set can be written as

$$(\pi_2, \mathcal{A}) = \begin{array}{c} \{ (\ t_0,\ \{\ s_2\ \}\ ), \\ (\ t_2,\ \{\ s_1, s_2, s_3\ \}\ ), \\ (\ t_3,\ \{\ s_1, s_4, s_5\ \}\ )\ \}, \end{array} \tag{4. 11}$$

Then $(\ (\ \pi_1, \pi_2\ ),\ \mathcal{A}\ )$ is a DFSS.
To consider another DFSS, take $\mathcal{B} = \{t_2, t_3\} \subset \mathcal{T}$, where
$\Omega_1 : \mathcal{B} \rightarrow P(\Im)$ is a mapping from $\mathcal{B}$ to $P(\Im)$, then the soft set can be written as

$$(\Omega_1, \mathcal{B}) = \{\ (\ t_2,\ \{\ s_1, s_3\ \}\ ),\ (\ t_3,\ \{\ s_4\ \}\ )\ \}, \tag{4. 12}$$

and $\Omega_2 : \mathcal{B} \rightarrow P(U)$ is a mapping from $\mathcal{B}$ to $P(U)$, then the soft set can be written as

$$(\Omega_2, \mathcal{B}) = \begin{array}{c} \{ (\ t_2,\ \{\ s_1, s_2\ \}\ ), \\ (\ t_3,\ \{\ s_1, s_5\ \}\ )\ \}, \end{array} \tag{4. 13}$$

Then $(\ (\ \Omega_1, \Omega_2\ ),\ \mathcal{B}\ )$ is a double framed soft set. Clearly, $(\ (\ \Omega_1, \Omega_2\ ),\ \mathcal{B}\ )$ is a double framed soft subset of $(\ (\ \pi_1, \pi_2\ ),\ \mathcal{A}\ )$. Therefore

$$\begin{array}{c} (\ (\ \pi_1, \pi_2\ ),\ \mathcal{A}\ )\ \cup\ (\ (\ \Omega_1, \Omega_2\ ),\ \mathcal{B}\ ) = \\ (\ (\ \varphi, \phi\ ),\ \mathcal{A} \cup \mathcal{B}\ ) \end{array} \tag{4. 14}$$

where

$$(\varphi, \mathcal{A} \cup \mathcal{B}) = \begin{array}{c} \{ (\ t_0,\ \{\ s_1, s_2, s_3, s_4, s_5\ \}\ ), \\ (\ t_2,\ \{\ s_1, s_3, s_4\ \}\ ), \\ (\ t_3,\ \{\ s_4, s_5\ \}\ )\ \}, \end{array} \tag{4. 15}$$

and

$$(\phi, \mathcal{A} \cup \mathcal{B}) = \begin{array}{c} \{ (\ t_0,\ \{\ h_2\ \}\ ), \\ (\ t_2,\ \{\ h_1, h_2, h_3\ \}\ ), \\ (\ t_3,\ \{\ h_1, h_4, h_5\ \}\ )\ \}, \end{array} \tag{4. 16}$$

$$\begin{array}{c} (\ (\ \pi_1, \pi_2\ ),\ \mathcal{A}\ )\ \cup\ (\ (\ \Omega_1, \Omega_2\ ),\ \mathcal{B}\ ) = \\ (\ (\ \varphi, \phi\ ),\ \mathcal{A} \cup \mathcal{B}\ ) = (\ (\ \pi_1, \pi_2\ ),\ \mathcal{A}\ ) \end{array} \tag{4. 17}$$

Here we observe that $(\,(\,\Omega_1, \Omega_2\,),\ \mathcal{B}\,)$ is double framed soft subset of $(\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,)$, then

$$(\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,)\ \cup\ (\,(\,\Omega_1, \Omega_2\,),\ \mathcal{B}\,) = \\ (\,(\,\varphi, \phi\,),\ \mathcal{A} \cup \mathcal{B}\,) = (\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,) \tag{4. 18}$$

which verifies the absorption law.

4.1.2. *Proposition 1.* For the double framed soft sets the following properties hold true.

(1) Union of two double soft sets is commutative.
i.e., $(\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,) \cup (\,(\,\Omega_1, \Omega_2\,),\ \mathcal{B}\,) = (\,(\,\Omega_1, \Omega_2\,),\ \mathcal{B}\,) \cup (\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,)$
(2) Union is associative.
i.e., $[\,(\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,) \cup (\,(\,\Omega_1, \Omega_2\,),\ \mathcal{B}\,)\,] \cup (\,(\,\zeta_1, \zeta_2\,),\ \mathcal{C}\,) = (\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,) \cup [\,(\,(\,\Omega_1, \Omega_2\,),\ \mathcal{B}\,) \cup (\,(\,\zeta_1, \zeta_2\,),\ \mathcal{C}\,)\,]$
(3) Union of double framed soft sets hold absorption law.
i.e., $(\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,) \cup (\,(\,\Omega_1, \Omega_2\,),\ \mathcal{B}\,) = (\,(\,\varphi, \phi\,),\ \mathcal{A} \cup \mathcal{B}\,) = (\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,)$.

4.2. **Intersection of Double Framed Soft Sets.** Let $(\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,)$ and $(\,(\,\Omega_1, \Omega_2\,),\ \mathcal{B}\,)$ be two soft sets, then their intersection is denoted and defined by

$$(\,(\,\pi_1, \pi_2\,),\ \mathcal{A}\,) \cap (\,(\,\Omega_1, \Omega_2\,),\ \mathcal{B}\,) = (\,(\,\phi, \varphi\,),\ \mathcal{A} \cap \mathcal{B}\,)$$

where

$$\phi(t) = \pi_1(t)\ \cap\ \Omega_1(t) \qquad \forall\ \ t \in \mathcal{A} \cap \mathcal{B} \tag{4. 19}$$

$$\varphi(t) = \pi_2(t)\ \cap\ \Omega_2(t) \qquad \forall\ \ t \in \mathcal{A} \cap \mathcal{B} \tag{4. 20}$$

4.2.1. *Example.* Consider that there are five houses in the initial universe set $\Im$ given by $\Im = \{s_1, s_2, s_3, s_4, s_5, \}$ and $\mathcal{T} = \{t_0, t_1, t_2, t_3\}$ be a set of parameters, where
t_0 represents ‘ beautiful ′,
t_1 represents ‘ Cheap ′,
t_2 represents ‘ in good location ′,
t_3 represents ‘ in green surroundings ′,
Assume that, $\mathcal{A} = \{t_0, t_2, t_3\} \subset \mathcal{T}$
where $\pi_1 : \mathcal{A} \to P(\Im)$ such that

$$(\pi_1, \mathcal{A}) = \ \ \{\,(\,t_0,\ \{\,s_1, s_2, s_3, s_4, s_5\,\}\,), \\ (\,t_2,\ \{\,s_1, s_3, s_4\,\}\,), \\ (\,t3,\ \{\,s_4, s_5\,\}\,)\,\}, \tag{4. 21}$$

where $\pi_2 : \mathcal{A} \to P(U)$ is a mapping, then the soft set given by

$$(\pi_2, \mathcal{A}) = \quad \{ (t_0, \{ s_2 \}), \\ (t_2, \{ s_1, s_2, s_3 \}), \\ (t_3, \{ s_1, s_4, s_5 \}) \}, \tag{4. 22}$$

Then $((\pi_1, \pi_2), \mathcal{A})$ is a DFSS. Consider another DFSS, take $\mathcal{B} = \{t_2, t_3\} \subset E$, where

$\Omega_1 : \mathcal{B} \to P(U)$ is a mapping, then the soft set can be written as

$$(\Omega_1, \mathcal{B}) = \{ (t_2, \{ s_1, s_3 \}), (t_3, \{ s_4 \}) \}, \tag{4. 23}$$

and $\Omega_2 : \mathcal{B} \to P(U)$ is a mapping, then the soft set can be written as

$$(\Omega_2, \mathcal{B}) = \{ (t_2, \{ s_1, s_2 \}), (t_3, \{ s_1, s_5 \}) \},$$

Then $((\Omega_1, \Omega_2), \mathcal{B})$ is a DFSS. Clearly, $((\Omega_1, \Omega_2), \mathcal{B})$ is a double framed soft subset of $((\pi_1, \pi_2), \mathcal{A})$. Therefore

$$((\pi_1, \pi_2), \mathcal{A}) \cap ((\Omega_1, \Omega_2), \mathcal{B}) = ((\phi, \varphi), \mathcal{A} \cap \mathcal{B})$$

where

$$(\phi, \mathcal{A} \cap \mathcal{B}) = \{ (t_2, \{ s_1, s_3 \}), (t_3, \{ s_4 \}) \},$$

and

$$(\varphi, \mathcal{A} \cap \mathcal{B}) = \{ (t_2, \{ s_1, s_2 \}), (t_3, \{ s_1, s_5 \}) \},$$

Clearly,

$$((\pi_1, \pi_2), \mathcal{A}) \cap ((\Omega_1, \Omega_2), \mathcal{B}) = \\ ((\phi, \varphi), \mathcal{A} \cap \mathcal{B}) = ((\pi_1, \pi_2), \mathcal{A}) \tag{4. 24}$$

which is law of absorption for intersection of double framed soft sets.

4.2.2. *Proposition 2.* For the double framed soft sets the following properties hold true.

(1) Intersection of two double soft sets is commutative.
i.e., $((\pi_1, \pi_2), \mathcal{A}) \cap ((\Omega_1, \Omega_2), \mathcal{B}) = ((\Omega_1, \Omega_2), \mathcal{B}) \cap ((\pi_1, \pi_2), \mathcal{A})$

(2) Intersection of double framed soft sets is associative.
i.e., $[((\pi_1, \pi_2), \mathcal{A}) \cap ((\Omega_1, \Omega_2), \mathcal{B})] \cap ((\zeta_1, \zeta_2), \mathcal{C}) = ((\pi_1, \pi_2), \mathcal{A}) \cap [((\Omega_1, \Omega_2), \mathcal{B}) \cap ((\zeta_1, \zeta_2), \mathcal{C})]$

(3) Intersection of two double framed soft sets also obey absorption law which is given by
$((\pi_1, \pi_2), \mathcal{A}) \cap ((\Omega_1, \Omega_2), \mathcal{B}) = ((\varphi, \phi), \mathcal{A} \cap \mathcal{B}) = ((\pi_1, \pi_2), \mathcal{A})$

4.3. **Difference of Double Framed Soft Sets.** Let $(\,(\,\pi_1,\pi_2\,),\mathcal{A})$ and $(\,(\,\Omega_1,\Omega_2\,),\mathcal{B}\,)$ be two DFSS with usual notations, then their difference set is defined as

$$(\,(\,\pi_1,\pi_2\,),\mathcal{A})\setminus(\,(\,\Omega_1,\Omega_2\,),\mathcal{B}\,)=(\,(\,\gamma_1,\gamma_2\,),\mathcal{A}\,)$$

where

$$\gamma_i(t)=\begin{cases}\pi_i(t), & \text{if } t\in\mathcal{A}\setminus\mathcal{B}\\ \pi_i(t)\setminus\Omega_i(t), & \text{if } t\in\mathcal{A}\cap\mathcal{B}.\end{cases}\quad i=1,2.$$

4.3.1. *Proposition 3.* Let $\Im$ be universe of discourse and $\mathcal{T}$ be set of parameters and further let $(\,(\,\pi_1,\pi_2\,),\mathcal{T})$ and $(\,(\,\Omega_1,\Omega_2\,),\mathcal{T}\,)$ be two double framed soft sets, where $\pi_i:\mathcal{T}\to P(\Im)$ and $\Omega_i:\mathcal{T}\to P(\Im)$ for $i=1,2$, then

1. $\{\,(\,(\,\pi_1,\pi_2\,),\ \mathcal{T})\ \cup\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)\,\}^c=$
$(\,(\,\pi_1,\pi_2\,),\ \mathcal{T}\,)^c\ \cap\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)^c$

2. $\{\,(\,(\,\pi_1,\pi_2\,),\ \mathcal{T})\ \cap\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)\,\}^c=$
$(\,(\,\pi_1,\pi_2\,),\ \mathcal{T}\,)^c\ \cup\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)^c$

Proof 1:
L.H.S= $\{\,(\,(\,\pi_1,\pi_2\,),\ \mathcal{T})\ \cup\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)\,\}^c$
$=(\,(\,\gamma_1,\gamma_2\,),\mathcal{T}\,)^c$, where $\gamma_i(t)=\pi_i(t)\cup\Omega_i(t),\quad\forall\quad t\in\mathcal{T},\quad i=1,2.$
By definition of the union of DFSS
$=(\,(\,\gamma_1^c,\gamma_2^c\,),\mathcal{T}\,)$, by definition of relative complement
$=(\,(\,(\,\pi_1\cup\Omega_1)^c,(\,\pi_2\cup\Omega_2\,)^c\,),\mathcal{T}\,)$,
$=(\,(\,\pi_1^c,\pi_2^c\,),E\,)\cap(\,(\,\Omega_1^c,\ \Omega_2^c\,),\mathcal{T}\,)$, By definition of relative complement of DFSS,
$=(\,(\,\pi_1,\pi_2\,),\mathcal{T}\,)^c\cap(\,(\,\Omega_1,\ \Omega_2\,),\mathcal{T}\,)^c$.
Hence
$\{\,(\,(\,\pi_1,\pi_2\,),\ \mathcal{T})\ \cup\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)\,\}^c=(\,(\,\pi_1,\pi_2\,),\ \mathcal{T}\,)^c\ \cap\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)^c$
proof 2:
$\{\,(\,(\,\pi_1,\pi_2\,),\ \mathcal{T})\ \cap\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)\,\}^c=(\,(\,\pi_1,\pi_2\,),\ \mathcal{T}\,)^c\ \cup\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)^c$
L.H.S= $\{\,(\,(\,\pi_1,\pi_2\,),\ \mathcal{T})\ \cap\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)\,\}^c$
Let $\pi_i(t)\cap\Omega_i(t)=\gamma_i(t)\quad\forall\quad t\in\mathcal{T}\quad\wedge\quad i=1,2.$
$=(\,(\,\gamma_1,\gamma_2\,),\mathcal{T}\,)^c$
$=(\,(\,\gamma_1^c,\gamma_2^c\,),\mathcal{T}\,)$, by definition of relative complement of DFSS,
$=(\,(\,(\,\pi_1\cap\Omega_1)^c,(\,\pi_2\cap\Omega_2\,)^c\,),\mathcal{T}\,)$,
$=(\,(\,\pi_1^c\cup\Omega_1^c\ ,\ \pi_2^c\cup\Omega_2^c\,),\mathcal{T}\,)$
$=(\,(\,\pi_1^c,\pi_2^c\,),\mathcal{T}\,)\cup(\,(\,\Omega_1^c,\ \Omega_2^c\,),\mathcal{T}\,)$,
$=(\,(\,\pi_1,\pi_2\,),\mathcal{T}\,)^c\cup(\,(\,\Omega_1,\ \Omega_2\,),\mathcal{T}\,)^c$
Hence
$\{\,(\,(\,\pi_1,\pi_2\,),\ \mathcal{T})\ \cap\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)\,\}^c=(\,(\,\pi_1,\pi_2\,),\ \mathcal{T}\,)^c\ \cup\ (\,(\,\Omega_1,\Omega_2\,),\ \mathcal{T}\,)^c$.

4.4. **Complement of a Double Framed Soft Set.** The complement of a double framed soft set $(\ (\ \pi, \Omega\ ), \mathcal{A})$ is written as $(\ (\ \pi, \Omega\ ), \mathcal{A})^c$ and is defined by $(\ (\ \pi, \Omega\ ), \mathcal{A})^c = (\ (\ \pi^c, \Omega^c), \neg\mathcal{A})$, such that

A mapping $\pi^c : \neg\mathcal{A} \rightarrow P(U)$, then $\pi^c(t) = U - \pi(t), \quad \forall \ \ t \in \neg\mathcal{A}$
A mapping $\Omega^c : \neg\mathcal{A} \rightarrow P(U)$, then $\Omega^c(t) = U - \Omega(t), \quad \forall \ \ t \in \neg\mathcal{A}$
Clearly, $(\ (\ (\ \pi, \Omega), \mathcal{A}\ )^c\ )^c = (\ (\ \pi, \Omega), \mathcal{A})$

4.5. **Relative Complement of a Double Framed Soft Set.** The relative complement of a DFSS $(\ (\ \pi, \Omega\ ), \mathcal{A})$ is written as $(\ (\ \pi, \Omega\ ), \mathcal{A})^r$ and is defined by $(\ (\ \pi, \Omega\ ), \mathcal{A})^r = (\ (\ \pi^r, \beta^r), \mathcal{A})$, such that

$\pi^r : \mathcal{A} \rightarrow P(U)$ is a mapping, given by $\pi^r(t) = U - \pi(t), \quad \forall \ \ t \in \mathcal{A}$

$\Omega^r : \mathcal{A} \rightarrow P(U)$ is a mapping, given by $\Omega^r(t) = U - \Omega(t), \quad \forall \ \ t \in \mathcal{A}$

Clearly, $(\ (\ (\ \pi, \Omega), \mathcal{A}\ )^r\ )^r = (\ (\ \pi, \Omega), \mathcal{A})$

After sharing double frame in details, Following is its extended form introduced as triple framed soft sets.

5. Triple Framed Soft Sets (TFSS)

This content is further description of Triple Framed soft sets. by using similar patterns, all the aggregative operations have been transformed in to triple and so on up to N framed soft sets.

Let E be set of parameters and $\Im$ be universal set and $\mathcal{A} \subset \mathcal{T}$ then $(\ (\ \pi_1, \pi_2, \pi_3\ ), \mathcal{A})$ is called a TFSS, where $\pi_i : \mathcal{A} \rightarrow P(\Im), \qquad i = 1, 2, 3.$

Remark:

If any one of these three mappings is null soft set then triple framed soft set becomes a double framed set. If any two of these mappings are null soft sets then this triple framed soft set becomes a soft set.

6. N-Framed Soft Sets (NFSS)

Let $\Im$ be the universe of discourse and $\mathcal{T}$ be set of parameters and further $\mathcal{A} \subset \mathcal{T}$, then $(\ (\ \pi_1, \pi_2, \pi_3, \cdots, \pi_n\ ),\ \mathcal{A})$ is called n-framed soft set, where $\pi_i : \mathcal{A} \rightarrow P(\Im), \quad i = 1, 2, 3, \cdots, n.$

7. Example of N-Framed Soft Set

As an example of n-framed soft set, two moves of chess game have beeb shared in which each move all the possibilities for all the pieces will be soft set, double framed soft set, triple framed soft and finally n-framed soft set . The following are the assumptions for the example

(1) We are listing the moves of white pieces.
(2) We label each box by an order pair.
(3) We start 1 to 8 from left to right and from bottom to top.
(4) We will set pieces of chess as a set of parameters.

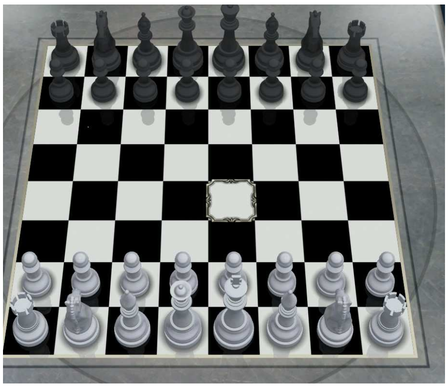

FIGURE 1. Chess Game

(5) White will choose first move.

Let us consider a set $\Im = \mathcal{A} \times \mathcal{A}$ as a universal set where, $\mathcal{A} = \{1, 2, 3, \cdots, 8\}$, and each member of $\Im$ represents a square box on chess board. Let's consider another set consisting of set of pieces of chess (which will serve the purpose of set of parameters) $\mathcal{T} = \{t_1, t_2, t_3, \cdots, t_{16}\}$ and let B be subset of set $\mathcal{T}$ given as $\mathcal{B} = \{t_1, t_2, t_3, \cdots, t_{16}\}$

where

t_1 is rook placed in box $(1, 1)$ left of queen
t_2 is a knight placed in box $(2, 1)$ left of the queen
t_3 is a bishop placed in box $(3, 1)$ left of the queen
t_4 is a queen placed in box $(4, 1)$
t_5 is the king placed in box $(5, 1)$ right side of the queen
t_6 is the bishop placed in box $(6, 1)$ right side of the queen
t_7 is the knight placed in the box labeled $(7, 1)$ right side of the Queen
t_8 is the rook placed in the box labeled as $(8, 1)$ right side of the Queen
t_9 is the pawn placed in the box labeled as $(1, 2)$
t_{10} is the pawn place in the box labeled as $(2, 2)$
t_{11} is the pawn placed in the box labeled as $(3, 2)$
t_{12} is the pawn placed in the box labeled as $(4, 2)$
t_{13} is the pawn placed in the box labeled as $(5, 2)$
t_{14} is the pawn placed in the box labeled as $(6, 2)$
t_{15} is the pawn placed in the box labeled as $(7, 2)$
t_{16} is the pawn placed in the box labeled as $(8, 2)$

and are shown in their original position in the chess as follows

TABLE 1. Chess game position's table

(1,8) t_1	(2,8) t_2	(3,8) t_3	(4,8) t_4	(5,8) t_5	(6,8) t_6	(7,8) t_7	(8,8) t_8
(1,7) t_9	(2,7) t_{10}	(3,7) t_{11}	(4,7) t_{12}	(5,7) t_{13}	(6,7) t_{14}	(7,7) t_{15}	(8,7) t_{16}
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)	(7,6)	(8,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)	(7,5)	(8,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)	(7,4)	(8,4)
(1,3) t_2	(2,3)	(3,3) t_2	(4,3)	(5,3)	(6,3) t_7	(7,3)	(8,3) t_7
(1,2) t_9	(2,2) t_{10}	(3,2) t_{11}	(4,2) t_{12}	(5,2) t_{13}	(6,2) t_{14}	(7,2) t_{15}	(8,2) t_{16}
(1,1) t_1	(2,1) t_2	(3,1) t_3	(4,1) t_4	(5,1) t_5	(6,1) t_6	(7,1) t_7	(8,1) t_8

$m_1, m_2, \cdots, m_n$, where $n \in N$ be the mappings which are defined as:
$m_i : \mathcal{B} \to P(\Im)$, $1 \leq i \leq n$ and $i \in N$, which can be given by the algorithm as follow.
$m_1(t_i) = \varphi, t_i \in \mathcal{B}$, where $i \neq 2, i \neq 7$, for $i \leq 8$
$m_1(t_2) = \{ (1,3), (3,3) \}$
$m_1(t_7) = \{ (6,3), (8,3) \}$
$m_1(t_{8+i}) = \{ (i,3), (i,4) \}, \quad i = 1, 2, \cdots, 8$
Hence $(m_1, \mathcal{B})$ is a soft set, which exhibit first move options for the player playing with white pieces. In this example, if we take a set $\mathcal{A} = \{ t_1, t_3, t_4, t_5, t_6, t_8 \}$, then $(m_i, \mathcal{A})$ is a null soft set.
Suppose that we move the first move as (which is not still moved just stopped to be moved.) $m_1(t_{13}) = \{ (5,4) \}$.
As we are intended to play king pawn type of opening of the game. Then the mapping $m_2 : \mathcal{B} \to P(\Im)$ is again a soft set representing the options for the players with white pieces.
Shown in the table below the other choices.
$m_2(t_i) = \phi$, where $i = 1, 3, 8$
$m_2(t_2) = \{ (1,3), (3,3) \}$
$m_2(t_7) = \{ (6,3), (8,3) \}$
$m_2(t_{8+i}) = \{ (i,3), (i,4) \}, \quad i = 1, 2, 3, 4, 6, 7, 8.$
$m_2(t_4) = \{ (i+3, i) \}, \quad i = 2, 3, 4, 5$
$m_2(t_6) = \{ (7-i, i) \}, \quad i = 2, 3, 4, 5$
$m_2(t_5) = \{ (5,2) \}$
Here $((m_1, m_2), \mathcal{B})$ is a double framed soft set. However if we take $\mathcal{C} = \{ t_1, t_3, t_8 \}$ and $m_1 : \mathcal{C} \to P(U)$ and $m_2 : \mathcal{C} \to P(\Im)$ be two mappings defined above, then $((m_1, m_2), \mathcal{A})$ is an example of double framed null soft set.
If we select another move as $m_2(t_7) = \{(6,3)\}$.

TABLE 2. Chess game position's table

(1,8) t_1	(2,8) t_2	(3,8) t_3	(4,8) t_4	(5,8) t_5	(6,8) t_6	(7,8) t_7	(8,8) t_8
(1,7) t_9	(2,7) t_{10}	(3,7) t_{11}	(4,7) t_{12}	(5,7) t_{13}	(6,7) t_{14}	(7,7) t_{15}	(8,7) t_{16}
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)	(7,6)	(8,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)	(7,5)	(8,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4) t_{13}	(6,4)	(7,4)	(8,4)
(1,3)	(2,3)	(3,3) t_2	(4,3)	(5,3)	(6,3)	(7,3)	(8,3) t_7
(1,2) t_9	(2,2) t_{10}	(3,2) t_{11}	(4,2) t_{12}	(5,2) t_{13}	(6,2) t_{14}	(7,2) t_{15}	(8,2) t_{16}
(1,1) t_1	(2,1) t_2	(3,1) t_3	(4,1) t_4	(5,1) t_5	(6,1) t_6	(7,1) t_7	(8,1) t_8

TABLE 3. Chess game position's table

(1,8) t_1	(2,8) t_2	(3,8) t_3	(4,8) t_4	(5,8) t_5	(6,8) t_6	(7,8) t_7	(8,8) t_8
(1,7) t_9	(2,7) t_{10}	(3,7) t_{11}	(4,7) t_{12}	(5,7) t_{13}	(6,7) t_{14}	(7,7) t_{15}	(8,7) t_{16}
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)	(7,6)	(8,6)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)	(7,5)	(8,5)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4) t_{13}	(6,4)	(7,4)	(8,4)
(1,3)	(2,3)	(3,3) t_2	(4,3)	(5,3)	(6,3) t_7	(7,3)	(8,3)
(1,2) t_9	(2,2) t_{10}	(3,2) t_{11}	(4,2) t_{12}	(5,2) t_{13}	(6,2) t_{14}	(7,2) t_{15}	(8,2) t_{16}
(1,1) t_1	(2,1) t_2	(3,1) t_3	(4,1) t_4	(5,1) t_5	(6,1) t_6	(7,1) t_7	(8,1) t_8

Now $m_3 : \mathcal{B} \to P(\Im)$ be another mapping which can be defined the way quite similar to the above mappings. Considering three moves before moving will be an example of TFSS, denoted by $((m_1, m_2, m_3), \mathcal{B})$.
we can also construct N-framed soft set showing the process thinking of n moves before selecting the first move of chess board. One can make N-framed soft set for the move

options for the black pieces. Interesting thing is that the person who has moved his piece; will have null soft set, while the other player will have non-empty soft sets.

8. Conclusion

This paper provides the NFSS by considering different problems that contain multiple functions against same parameters simultaneously. Different aggregative operations of soft sets have also been defined which are helpful for making different decisions to manage uncertainties and to minimize the chances of failures. Double-framed, triple framed and in general NFSS have also been shared with the help of several applications. This study takes the parameterized family of NFSS and defines the aggregative operators. Relevant examples are also solved by utilizing the aggregative operators to develop a clear understanding of the phenomenon.

Future research will focus on applying the idea/results to N-framed fuzzy soft sets, N-framed interval valued soft sets, N-framed intuitionistic soft sets and the related soft algebraic structures.

References

[1] M.I. Ali, F. Feng, X. Liu, W.K. Min, *On some new operations in soft set theory*, Computers and Mathematics with Appl., **57**, No.9 (2009) 1547-1553.

[2] N. Cagman, S. Enginoglu, *Soft matrix theory and its decision making*, Computers and Mathematics with Appl., **59** (2010) 3308-3314.

[3] N. Cagman, S. Enginoglu, *Soft set theory and uni-int decision making*, Europian Journal of Operational Research, **207** (2010) 848 - 855.

[4] Chandrasekhar U., Mathur S. *Decision Making Using Fuzzy Soft Set Inference System*, Proceedings of the 3rd International Symposium on Big Data and Cloud Computing Challenges (ISBCC - 16'), Smart Innovation, Systems and Technologies, Springer, **49**(2016).

[5] T. Doenovic, D. Rakic, B. Caric, S. Radenovic, *Multivalued generalizations of fixed point results in fuzzy metric spaces*, Nonlinear Analysis: Modelling and Control, **21**,No.2, 211-222.

[6] Y. B. Jun, S. S. Ahn, *Double-Framed Soft Sets with Applications in BCK/BCI-Algebras*, Journal of Applied Mathematics,**2012** 1-15.

[7] Ljubomir Ciric, *Some Recent Results in Metrical Fixed Point Theory*,Journal of Fixed Point Theory and Applications,**2**, No.1(2003)195-207.

[8] P.K. Maji, R. Biswas, A.R. Roy, *Soft set theory*, Computers and Mathematics with Applications, **45** (2003) 555-562.

[9] P.K. Maji, A.R. Roy, R. Biswas, *An application of soft sets in a decision making problem*, Computers and Mathematics with Applications, **44** (2002) 10771083.

[10] D. Molodtsov, *Soft set theory-first results*, Computers and Mathematics with Applications, **37** (1999) 1931.

[11] I.A. Onyeozili, T.M. Gwary, *A study of the fundamentals of soft set theory*, International journal of scientific and technology research, **3** (2014).

[12] Z. Pawlak, *Rough sets*, International Journal of Information and Computer Sciences, **11**(1982) 341356.

[13] Z. Pawlak, A. Skowron, *Rudiments of rough sets*, Information Sciences, **177** (2007) 327.

[14] M. Riaz and K. Naeem, *Measurable Soft Mappings*, Punjab Univ. j. math., **48**,(2)(2016), 19-34.

[15] M. Riaz, M. R. Hashmi, *Fuzzy parameterized fuzzy soft compact spaces with decision-making*, Punjab Univ. j. math., 50,(2)(2018) 131-145.

[16] M. Riaz, M. R. Hashmi, *Fixed points of fuzzy neutrosophic soft mapping with decision-making*, Fixed point theory and applications, **7**(2018) 1-10.

[17] M. Riaz, F. Samrandache, A. Firdous and F. Fakhar, *On Soft Rough Topology with Multi-Attribute Group Decision Making*,Mathematics, **7**, No.67(2019. doi:10.3390/math7010067.

[18] M. Riaz, B. Davvaz, A. Firdous and F. Fakhar, *Novel Concepts of Soft Rough Set Topology with Applications*, Journal of Intelligent & Fuzzy Systems, **36**, No.4(2019), 3579-3590.

[19] M. Riaz, N. Çağman, I. Zareef and M. Aslam, *N-Soft Topology and its Applications to Multi-Criteria Group Decision Making*, Journal of Intelligent & Fuzzy Systems (In Press) (2019). DOI:10.3233/JIFS-182919

[20] M. Riaz and S. T. Tehrim, *Certain properties of bipolar fuzzy soft topology via Q-neighborhood*, Punjab Univ. j. math. **51**, No.3(2019) 113-131.

[21] M. Riaz and S. T. Tehrim, *Multi-attribute group decision making based cubic bipolar fuzzy information using averaging aggregation operators*, Journal of Intelligent & Fuzzy Systems (In Press) (2019).

[22] M. Riaz, Saqlain.M. and Saeed.M. *Application of Generalized Fuzzy TOPSIS in Decision Making for Neutrosophic Soft set to Predict the Champion of FIFA 2018: A Mathematical Analysis*, Punjab Univ. j. math., **51,** No.6 (2019) 111-126.

[23] M. Saqlain, M. Saeed, M. R. Ahmad, F. Smarandache, *Generalization of TOPSIS for Neutrosophic Hypersoft set using Accuracy Function and its Application*, Neutrosophic Sets and Systems (NSS), **27,**(2019)131-137.

[24] M. Saqlain., N. Jafar. and A. Riffat.,*Smart phone selection by consumers' in Pakistan: FMCGDM fuzzy multiple criteria group decision making approach*, Gomal University Journal of Research, **34,** No.1 (2019), 27-31.

[25] T.R. Sooraj, Mohanty R.K., Tripathy B.K. *Fuzzy Soft Set Theory and Its Application in Group Decision Making*, Advances in Intelligent Systems and Computing, **452** (2016).

[26] D. Singh, I.A. Onyeozili, *Notes on soft matrices operations*, ARPN Journal of Science and Technology,**2,** No.9 (2012) 861-869.

[27] B.K. Tripathy , Arun K.R. *A new approach to soft sets, soft multisets and their properties*. International Journal of Reasoning-Based Intelligent Systems, **7,** No.3-4 (2015) 244-253.

[28] Tripathy B.K., Sooraj T.R., Mohanty R.K. *A New Approach to Fuzzy Soft Set Theory and Its Application in Decision Making*. Advances in Intelligent Systems and Computing, **411** (2016).

[29] Vesna Todorcevic, *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*, Springer Nature Switzerland AG (2019).

[30] L.A. Zadeh, *Fuzzy sets*, Information Control, **8** (1965) 338353.

[31] L.A. Zadeh, *Toward a generalized theory of uncertainty (GTU)-an outline*, Information Sciences, **172** (2005) 140.

[32] P. Zhu, W. Qiaoyan, *Operations on soft sets revisted*, Journal of Applied mathematics,**2013** 105752.