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Statistical analysis of subdivision schemes in terms of probability distribution

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Abstract. The objective of this article is to develop the link between probability distribution theory and subdivision schemes. The distribution theory is a field of statistics and subdivision is a field of Computer Aided Geometric Design, both are independent fields. The analysis of subdivision schemes using probability distribution theory is presented. In our analysis, we explore the important properties such as mean, variance, moments about the origin, moments about mean, measures of skewness and measures of kurtosis of subdivision schemes presented in Chinese Annals of Mathematics, Series B (8(5): 1077-1092, 2017).

AMS (MOS) Subject Classification Codes: 65D17, 65D15, 65D10, 60E05, 62H10 .
Key Words: Probability distribution; approximating subdivision schemes; B-spline basis function; mean; variance; measures of skewness; measures of kurtosis; Laurent polynomial.

1. Introduction

Initially, subdivision schemes were introduced for the modeling of curves and surfaces. During the last decade the subdivision schemes collaborate with different fields. Hed et al. [2] presented an algorithm for subdivision schemes. The hierarchical structures of music compositions were also presented in their work. Brakhage [3] used subdivision schemes for grid generation and grid conversion. Charina et al. [4] presented and analyzed new grid transformer and grid operator with subdivision schemes. Their operator can be used for the solution of differential equations. They also presented a univariate binary and ternary subdivision schemes by using the grid transformer. Ejaz et al. [8] presented two collocation algorithms which are based on interpolating and approximating subdivision schemes for the solution of differential equations. They concluded that the algorithm based on approximating schemes give better results comparing to the interpolating schemes based collocation algorithm. Weimer and Warren [9] used subdivision schemes for the solution of partial differential equations.

Traditionally subdivision schemes are analyzed by emphasizing on the geometrical properties (i.e. order of approximation, polynomial reproduction, continuity and Hölder regularity) of the schemes. In the previous five years, the subdivision schemes collaborated with statistics for refining of data. Dyn et al. [5] presented least squares based subdivision schemes for noisy data, but these schemes cannot deal with data containing outliers. Mustafa et al. [6] introduced the iterative reweighted least squares based subdivision schemes to deal with outliers and impulsive noise. Mustafa [7] presented a numerical way for the model selection from the family of subdivision schemes. He has used the statistical tools such as training errors, curvature and residual sum of squares for the model selection. The authors in [5, 6, 7] emphasis more on numerical computing and less on analytical findings. In this paper, we present analytic proof of the statistical entities such as, mean, variance, moments about origin, moments about mean, measures of skewness and measures of kurtosis.

The shape preservation of a given data is an important topic in the field of data visualization. In this study, we investigate whether or not the shapes are preserved by the subdivision schemes while the data is negatively skewed, positively skewed and symmetrical etc.

We have used the statistical tool named probability distributions to achieve the goal. It is a mathematical function which provide the probability of occurrence of different possible outcomes in an experiment. Let $i \in \mathbb{N}_0$ be a discrete random variable then a mathematical function $f : i \to \mathbb{R}$ is called probability density function if it satisfies two conditions

- $f(i) \geq 0, \forall\, i,$
- $\sum_i f(i) = 1.$

The paper is structured as follows: In Section 2, we rewrite the family of schemes introduced by Mustafa [7]. Section 3 is devoted for the statistical analysis of the family of schemes. Section 4 is for numerical experiments. Conclusions are drawn in Sections 5.

2. A GENERALIZED FAMILY OF SCHEMES

In this paper, we consider the family of approximating schemes [7] and discuss its statistical properties. The properties of other existing schemes with positive mask in the literature will be explored similarly.
If m and n represent the complexity and arity of the schemes then the family of schemes is defined as:

$$\begin{cases} f_{ni}^{k+1} = \sum\limits_{j\in\mathbb{Z},k=0}^{m-1} a_{k,1} f_{i-j}^{k} \\ f_{ni+1}^{k+1} = \sum\limits_{j\in\mathbb{Z},k=0}^{m-1} a_{k,2} f_{i-j}^{k} \\ f_{ni+2}^{k+1} = \sum\limits_{j\in\mathbb{Z},k=0}^{m-1} a_{k,3} f_{i-j}^{k} \\ \quad\vdots \qquad\qquad \vdots \\ f_{ni+(n-3)}^{k+1} = \sum\limits_{j\in\mathbb{Z},k=0}^{m-1} a_{(m-1-k),3} f_{i-j}^{k} \\ f_{ni+(n-2)}^{k+1} = \sum\limits_{j\in\mathbb{Z},k=0}^{m-1} a_{(m-1-k),2} f_{i-j}^{k} \\ f_{ni+(n-1)}^{k+1} = \sum\limits_{j\in\mathbb{Z},k=0}^{m-1} a_{(m-1-k),1} f_{i-j}^{k} \end{cases} \tag{2. 1}$$

where the coefficients $a_{i,j}$ (also called mask) are defined as:

$$a_{i,j} = N_{0,m}\left((m-i) - \frac{(2n-\alpha)}{2n}\right), \quad i = 0, 1, 2, \cdots, m-1, \tag{2. 2}$$

where $(j,\alpha) = \{(1,1), (2,3)\cdots(\frac{n}{2}, n+1)$, for even integer n$\}$,
$(j,\alpha) = \{(1,1), (2,3)\cdots(\frac{n+1}{2}, n)$, for odd integer n$\}$,

$$N_{0,m}(t) = \frac{t}{m-1}N_{0,m-1}(t) + \frac{m-t}{m-1}N_{0,m-1}(t-1), \quad m \neq 1 \tag{2. 3}$$

and

$$N_{0,1}(t) = \begin{cases} 1 & \text{if} \quad t_0 \leq t \leq t_1, \\ 0 & \text{otherwise.} \end{cases}$$

The mask of the scheme plays a vital role to discuss the geometrical properties of the schemes. So we focus on the mask of the schemes to explore more characteristics of the schemes.

We can rewrite (2.2) in the form of probability density function $f : i \to \mathbb{R}$ defined as $f(i,n,m,\alpha) = a_{i,j} = N_{0,m}\left((m-i) - \frac{(2n-\alpha)}{2n}\right)$, where i is the discrete random variable. It satisfies the two basic conditions of the probability density function

- $f(i,n,m,\alpha) \geq 0 \; \forall \, i$,
- $\sum_i f(i,n,m,\alpha) = \sum_{i=0}^{m-1} a_{i,j} = 1$.

It means that the mask of the scheme can be considered a discrete probability density function. Which motivate us to explore the relation between the subdivision schemes and probability distributions.

3. Statistical analysis

In this section, we will discuss the important properties such as mean, variance, moment about origin, moment about mean, measure of skewness and measure of kurtosis of the subdivision scheme (2.1).

3.1. **Moments about origin.** Here we will present the r^{th} moments about the origin (i.e., ordinary moments) of the subdivision scheme. The zeroth moment is the total probability (i.e. one), the first moment about the origin is the mean of the subdivision scheme.

Theorem 3.2. *The mean of subdivision scheme* (2.1) *is* $\mu'_{1,m} = \frac{mn-(2n-\alpha)}{2n}$.

Proof. As we know that the mean of the distribution can be calculated as

$$\mu'_{1,m} = \sum_{i=0}^{m-1} i.a_{i,j}.$$

By substituting the value of $a_{i,j}$ from (2.2), we have

$$\mu'_{1,m} = \sum_{i=0}^{m-1} i.N_{0,m}\left((m-i) - \frac{(2n-\alpha)}{2n}\right).$$

Using (2.3), we get

$$\begin{aligned}\mu'_{1,m} &= \sum_{i=0}^{m-1} i.\Big\{\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big) + \Big(\frac{m-(m-i-\frac{2n-\alpha}{2n})}{m-1}\Big)\\ &\quad \times N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}-1\Big)\Big\}.\end{aligned}$$

This implies that

$$\begin{aligned}\mu'_{1,m} &= \sum_{i=0}^{m-1} i.\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big)\\ &\quad + \sum_{i=0}^{m-1} i.\Big(\frac{m-(m-i-\frac{2n-\alpha}{2n})}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}-1\Big).\end{aligned}$$

Expanding the first and last terms of the first and second summations respectively, we get

$$\begin{aligned}\mu'_{1,m} &= \sum_{i=1}^{m-1} i.\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big) + \sum_{i=0}^{m-2} i.\Big(\frac{i+\frac{2n-\alpha}{2n}}{m-1}\Big)\\ &\quad \times N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}-1\Big).\end{aligned}$$

Now replace i by $i-1$ in second summation

$$\begin{aligned}\mu'_{1,m} &= \sum_{i=1}^{m-1} i.\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big)\Big)\\ &\quad + \sum_{i=1}^{m-1} (i-1).\Big(\frac{i-1+\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-(i-1)-\frac{2n-\alpha}{2n}-1\Big)\Big).\end{aligned}$$

This implies that

$$\mu'_{1,m} = \sum_{i=1}^{m-1} i.\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big)\Big)$$

$$+\sum_{i=1}^{m-1}(i-1).\Big(\frac{i-1+\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big)\Big).$$

Further implies

$$\mu'_{1,m} \quad = \quad \sum_{i=1}^{m-1}\Big(i.\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}\Big)+(i-1).\Big(\frac{i-1+\frac{2n-\alpha}{2n}}{m-1}\Big)\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big)\Big).$$

Simplified form is

$$\mu'_{1,m} \quad = \quad \sum_{i=1}^{m-1}\Big(\frac{i(m-2)+\frac{\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big). \qquad (3.\ 4)$$

A further simplified form can be obtained by setting $m = 2, 3, \cdots$. By setting $m = 2$ in (3.4), we get $\mu'_{1,2} = \frac{2n-(2n-\alpha)}{2n}$. By setting $m = 3$ in (3.4), we get $\mu'_{1,3} = \frac{3n-(2n-\alpha)}{2n}$. Similarly in general, we have $\mu'_{1,m} = \frac{mn-(2n-\alpha)}{2n}$. This completes the proof. □

Remark 3.3. *We conclude the following from Figure 1:*

- *From Figure 1(a), the mean of lower arity subdivision schemes are greater than the mean of higher arity subdivision schemes.*
- *From Figure 1(b), the mean of higher complexity schemes are greater than the mean of lower complexity subdivision schemes.*
- *From Figure 1(c), in binary case, the mean of higher complexity schemes are greater than the mean of lower complexity subdivision schemes.*
- *From Figure 1(d), the mean of lower arity subdivision schemes are greater than the mean of higher arity schemes for fix complexity $m = 2$.*

Lemma 3.4. *The r^{th} moment of the subdivision scheme* (2.1) *is*

$$\mu'_{r,m} \quad = \quad \sum_{i=1}^{m-1}\Big(\frac{i^r(m-i-\frac{2n-\alpha}{2n})+(i-1)^r(i-\frac{\alpha}{2n})}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big).$$

Proof. As we know that the r^{th} moment about the origin of the distribution can be defined as

$$E(i^r) = \mu'_{r,m} = \sum_{i=0}^{m-1} i^r a_{i,j}. \qquad (3.\ 5)$$

By substituting the value of $a_{i,j}$ from (2.2), we have

$$\mu'_{r,m} = \sum_{i=0}^{m-1} i^r.N_{0,m}\left((m-i)-\frac{(2n-\alpha)}{2n}\right).$$

Using (2.3), we get

$$\mu'_{r,m} \quad = \quad \sum_{i=0}^{m-1} i^r.\Big\{\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big)+\Big(\frac{m-(m-i-\frac{2n-\alpha}{2n})}{m-1}\Big)$$

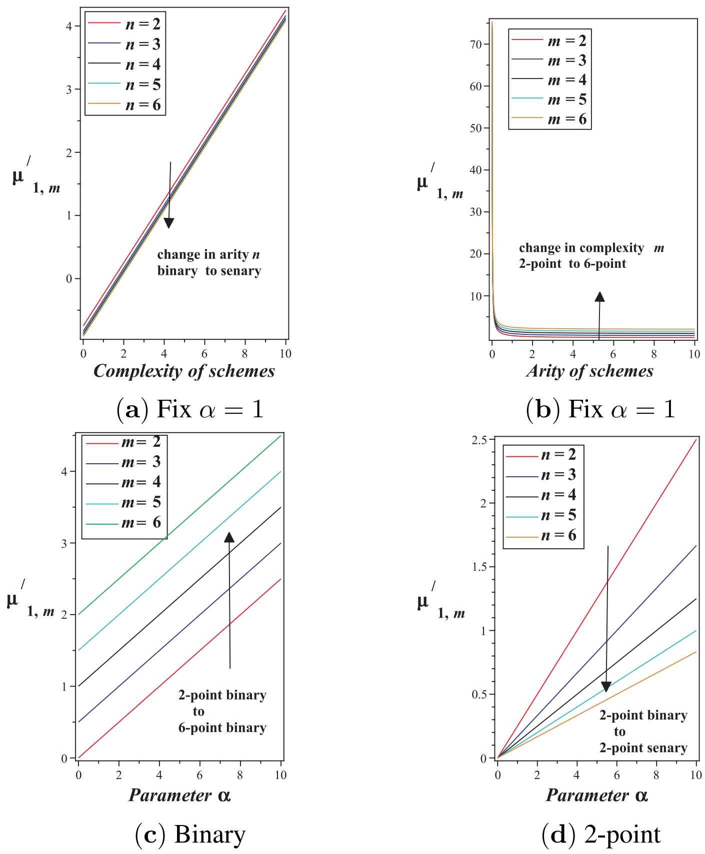


FIGURE 1. Plots of mean $\mu'_{1,m}$ for different values of m, n and α.

$$\times N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}-1\Big)\Big\}.$$

This implies that

$$\begin{aligned}\mu'_{r,m} &= \sum_{i=0}^{m-1} i^r.\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big)\\ &\quad+\sum_{i=0}^{m-1} i^r.\Big(\frac{m-(m-i-\frac{2n-\alpha}{2n})}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}-1\Big).\end{aligned}$$

Expanding first and last terms of the first and second summations respectively, we get

$$\mu'_{r,m} = \sum_{i=1}^{m-1} i^r.\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big)$$

$$+\sum_{i=0}^{m-2} i^r.\Big(\frac{i+\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}-1\Big).$$

Now replace i by $i-1$ in second summation

$$\begin{aligned}\mu'_{r,m} &= \sum_{i=1}^{m-1} i^r.\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big)\\ &+\sum_{i=1}^{m-1}(i-1)^r.\Big(\frac{i-1+\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-(i-1)-\frac{2n-\alpha}{2n}-1\Big).\end{aligned}$$

This implies

$$\begin{aligned}\mu'_{r,m} &= \sum_{i=1}^{m-1} i^r.\Big(\frac{m-i-\frac{2n-\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big)\\ &+\sum_{i=1}^{m-1}(i-1)^r.\Big(\frac{i-\frac{\alpha}{2n}}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big).\end{aligned}$$

Further implies

$$\mu'_{r,m}=\sum_{i=1}^{m-1}\Big(\frac{i^r(m-i-\frac{2n-\alpha}{2n})+(i-1)^r(i-\frac{\alpha}{2n})}{m-1}\Big)N_{0,m-1}\Big(m-i-\frac{2n-\alpha}{2n}\Big). \quad (3.\ 6)$$

This completes the proof. □

Lemma 3.5. *The r^{th} moment about the origin of 2-point subdivision scheme (2.1) is $\mu'_{r,2}=\frac{\alpha}{2n}$.*

Proof. By substituting $m=2$ in (3.6), we have

$$\mu'_{r,2}=\sum_{i=1}^{2-1}\Big(\frac{i^r(2-i-\frac{2n-\alpha}{2n})+(i-1)^r(i-\frac{\alpha}{2n})}{2-1}\Big)N_{0,2-1}\Big(2-i-\frac{2n-\alpha}{2n}\Big).$$

This implies

$$\mu'_{r,2}=\sum_{i=1}^{1}\Big(\frac{i^r(2-i-\frac{2n-\alpha}{2n})+(i-1)^r(i-\frac{\alpha}{2n})}{1}\Big)N_{0,1}\Big(2-i-\frac{2n-\alpha}{2n}\Big).$$

After simplification, we get

$$\mu'_{r,2}=\frac{\alpha}{2n}.$$

This completes the proof. □

Remark 3.6. *The Table 1 gives the presentation of r^{th} moments about the origin. From this table, we conclude that the r^{th} moments can be expressed in the form of $(m-1)th$ degree polynomial. i.e. $p(x)=a_{m-1}x^{m-1}+a_{m-2}x^{m-2}+\cdots+a_1x+a_0$ where $x=\frac{\alpha}{2n}$.*

TABLE 1. *Shows the general formulas of second, third and fourth moments about origin for $m = 2, 3, 4$ and 5.*

r	m	$\mu'_{r,m}$
2	2	$\mu'_{2,2} = \frac{\alpha}{2n}$
2	3	$\mu'_{2,3} = 2(\frac{\alpha}{2n})^2 + 2(\frac{\alpha}{2n}) + 1$
2	4	$\mu'_{2,4} = 3(\frac{\alpha}{2n})^2 + 6(\frac{\alpha}{2n}) + 4$
2	5	$\mu'_{2,5} = 4(\frac{\alpha}{2n})^2 + 12(\frac{\alpha}{2n}) + \frac{32}{3}$
3	2	$\mu'_{3,2} = \frac{\alpha}{2n}$
3	3	$\mu'_{3,3} = 3(\frac{\alpha}{2n})^2 + \frac{\alpha}{2n} + \frac{1}{2}$
3	4	$\mu'_{3,4} = (\frac{\alpha}{2n})^3 + 3(\frac{\alpha}{2n})^2 + 4(\frac{\alpha}{2n}) + 2$
3	5	$\mu'_{3,5} = (\frac{\alpha}{2n})^3 + \frac{9}{2}(\frac{\alpha}{2n})^2 + 8(\frac{\alpha}{2n}) + \frac{42}{8}$
4	2	$\mu'_{4,2} = \frac{\alpha}{2n}$
4	3	$\mu'_{4,3} = 7(\frac{\alpha}{2n})^2 + \frac{\alpha}{2n} + \frac{1}{2}$
4	4	$\mu'_{4,4} = 6(\frac{\alpha}{2n})^3 + 14(\frac{\alpha}{2n})^2 + 8(\frac{\alpha}{2n}) + \frac{40}{12}$
4	5	$\mu'_{4,5} = (\frac{\alpha}{2n})^4 + 6(\frac{\alpha}{2n})^3 + 16(\frac{\alpha}{2n})^2 + 21(\frac{\alpha}{2n}) + \frac{67}{6}$

3.7. **Measures of skewness and kurtosis.** In this subsection, we will discuss the moments about the mean (i.e., central moments) using the relation between moments about the origin [1]. It is well known from distribution theory that, first moment about the mean $\mu_{1,m}$ is always zero. Second, third and fourth moment about the mean can be expressed as

$$\mu_{2,m} = \mu'_{2,m} - (\mu'_{1,m})^2 \tag{3. 7}$$

$$\mu_{3,m} = \mu'_{3,m} - 3\mu'_{1,m}\mu'_{2,m} + 2(\mu'_{1,m})^3 \tag{3. 8}$$

$$\mu_{4,m} = \mu'_{4,m} - 4\mu'_{1,m}\mu'_{3,m} + 6(\mu'_{1,m})^2\mu'_2 - 3(\mu'_1)^4 \tag{3. 9}$$

respectively.
Here we will discuss the variance (i.e., 2nd central moment) of the subdivision schemes (2.1).

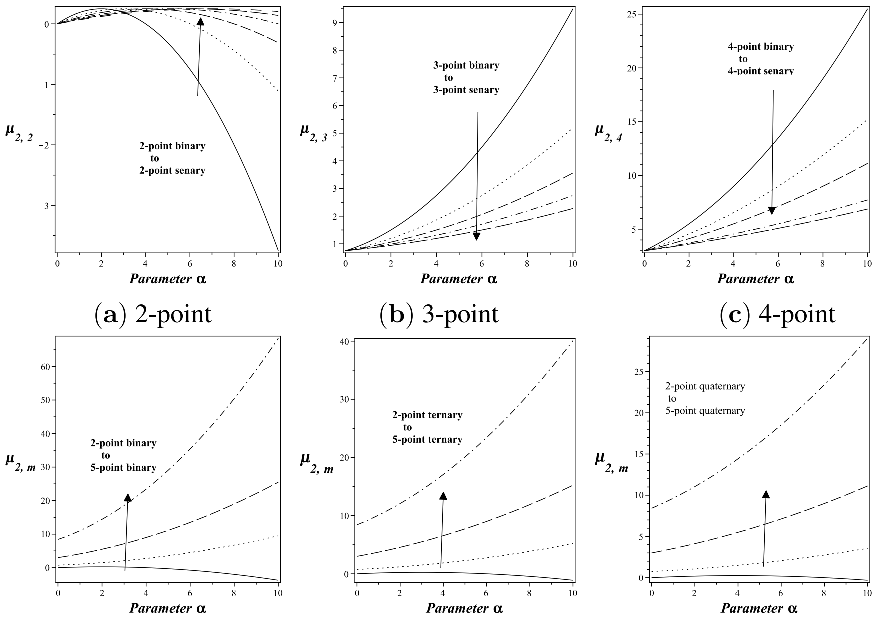
Proposition 3.8. *The 2^{nd} moment about the mean (i.e. variance) of the scheme against the different values of m are given below.*

- *For $m = 2$, $\mu_{2,2} = -(\frac{\alpha}{2n})^2 + \frac{\alpha}{2n}$.*
- *For $m = 3$, $\mu_{2,3} = (\frac{\alpha}{2n})^2 + \frac{\alpha}{2n} + \frac{3}{4}$.*
- *For $m = 4$, $\mu_{2,4} = 2(\frac{\alpha}{2n})^2 + 4(\frac{\alpha}{2n}) + 3$.*
- *For $m = 5$, $\mu_{2,5} = 6(\frac{\alpha}{2n})^2 + 9(\frac{\alpha}{2n}) + \frac{101}{12}$.*

Proof. We can easily verify the above results by substituting $m = 2, 3, 4$ and 5 in (3.7). □

Remark 3.9. *We conclude the following results from Figure 2:*

- *From Figure 2(a), the variance of higher arity subdivision schemes are greater than the variance of lower arity subdivision schemes for fix complexity $m = 2$.*
- *From Figures 2(b) and 2(c), the variance of lower arity subdivision schemes are greater than the variance of higher arity subdivision schemes for fix complexity $m = 3$ and $m = 4$.*



$(\mathbf{a})$ 2-point $(\mathbf{b})$ 3-point $(\mathbf{c})$ 4-point

$(\mathbf{d})$ Binary $(\mathbf{e})$ Ternary $(\mathbf{f})$ Quaternary

FIGURE 2. Plots of variance $\mu_{2,m}$ for different values of m, n and α.

- *From Figures 2(d), 2(e) and 2(f), the variance of higher complexity is greater than the variance of lower complexity subdivision schemes for fix arity: for binary, $n = 2$, for ternary, $n = 3$ and for quaternary, $n = 4$.*

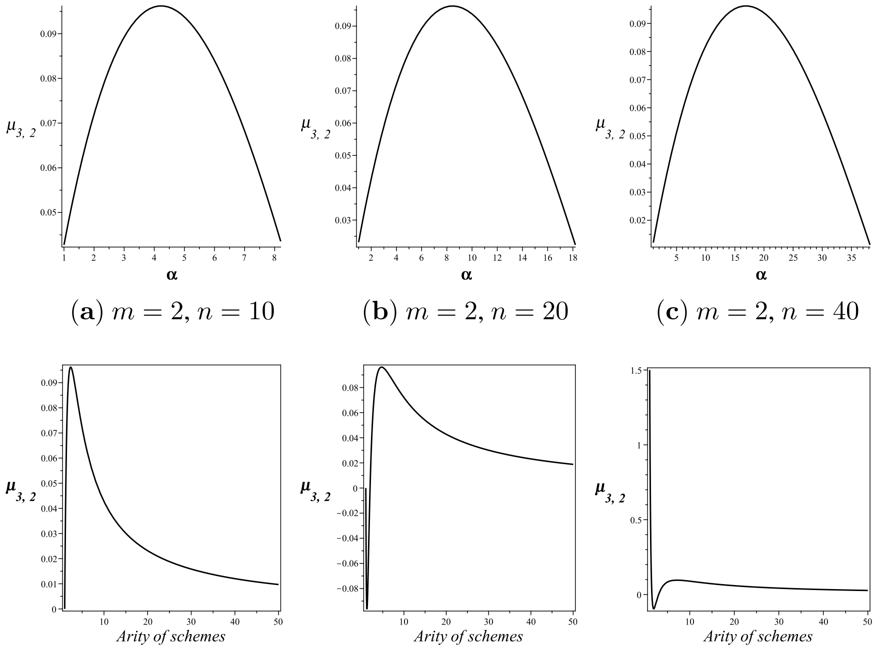
Skewed behavior: In this section, we discuss the skewed behavior of the subdivision scheme. The third moment about mean (i.e., third central moment) $\mu_{3,m}$ is helpful for the discussion of skewed behavior of the subdivision scheme. The subdivision scheme is negatively skewed, positively skewed and symmetrical (i.e., mean= mode= median) when $\mu_{3,m} < 0$, $\mu_{3,m} > 0$ and $\mu_{3,m} = 0$ respectively.

Proposition 3.10. *The 3^{rd} moments about the mean against the different values of m of the scheme are given below.*

- *For $m = 2$, $\mu_{3,2} = 2(\frac{\alpha}{2n})^3 - 3(\frac{\alpha}{2n})^2 + \frac{\alpha}{2n}$.*
- *For $m = 3$, $\mu_{3,3} = -(4(\frac{\alpha}{2n})^3 + 3(\frac{\alpha}{2n})^2 + 7(\frac{\alpha}{2n}) + \frac{3}{4})$.*
- *For $m = 4$, $\mu_{3,4} = -(6(\frac{\alpha}{2n})^3 + 18(\frac{\alpha}{2n})^2 + 20(\frac{\alpha}{2n}) + 8)$.*
- *For $m = 5$, $\mu_{3,5} = -(9(\frac{\alpha}{2n})^3 + \frac{81}{2}(\frac{\alpha}{2n})^2 + \frac{129}{2}(\frac{\alpha}{2n}) + 36)$*

Proof. We can easily verify the above results by substituting $m = 2, 3, 4$ and 5 in (3.5). □

Remark 3.11. *The Figure 3 shows the skewed behavior of the subdivision scheme. For fix complexity $m = 2$ (2-point), the limiting shape produced by the subdivision scheme depends on arity n and parameter α. We notice that:*



(**a**) $m = 2, n = 10$ (**b**) $m = 2, n = 20$ (**c**) $m = 2, n = 40$

(**d**) $m = 2, \alpha = 1$ (**e**) $m = 2, \alpha = 2$ (**f**) $m = 2, \alpha = 3$

FIGURE 3. Plots of 3^{rd} moment $\mu_{3,m}$ for 2-point, n-ary schemes.

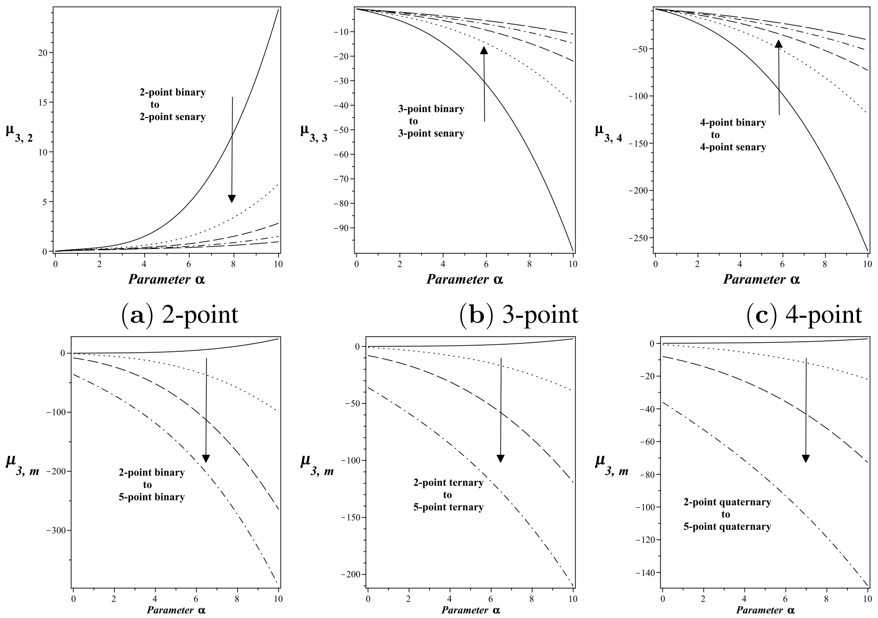
- *The limiting shape produced by the scheme is symmetrical when* $\alpha = n - \epsilon$ *and* $\epsilon = [1.5, 1.8]$. *(See Figure* 3*: (a), (b) (c)).*
- *The limiting shape is positively skewed for the parameter* $\alpha = 1$ *and arity* $n \geq 2$, *as arity increases the degree of right skewness increases. (See Figure* 3*: (d), (e) (f))*

Remark 3.12. *The Figure 4 also shows the skewed behavior of the scheme.*

- *From Figure 4(a), we see that as the arity of the 2-point schemes increases the shape of the limiting curves produced by these schemes are changing from positively to negatively skewed shapes.*
- *From Figures 4(b) and 4(c), we see that the 3-point and 4-point* n*-ary schemes produce the negatively skewed shapes.*
- *From Figures 4(d), 4(e) and 4(f), higher arity subdivision schemes produce the shapes which are less negatively skewed comparing to the shapes produced by low arity subdivision schemes for complexity* $(m = 2, 3 \cdots, 5)$.

Also $\mu_{3,m} < 0$ *for all* $m \geq 3$, $\alpha \geq 1$ *and* $n \geq 2$, *this implies that in general the subdivision scheme* (2.1) *produce the negatively skewed shapes.*

Proposition 3.13. *The* 4^{th} *moments about mean against the different values of* m *of the scheme are given below.*



$(\mathbf{a})$ 2-point $(\mathbf{b})$ 3-point $(\mathbf{c})$ 4-point

$(\mathbf{d})$ Binary $(\mathbf{e})$ Ternary $(\mathbf{f})$ Quaternary

FIGURE 4. Plots of 3^{rd} moment $\mu_{3,m}$ for different values of m, n and α.

- *For* $m = 2$, $\mu_{4,2} = -3(\frac{\alpha}{2n})^4 + 6(\frac{\alpha}{2n})^3 - 4(\frac{\alpha}{2n})^2 + \frac{\alpha}{2n}$.
- *For* $m = 3$, $\mu_{4,3} = 9(\frac{\alpha}{2n})^4 + 6(\frac{\alpha}{2n})^3 + \frac{54}{4}(\frac{\alpha}{2n})^2 + \frac{36}{8}(\frac{\alpha}{2n}) + \frac{13}{16}$.
- *For* $m = 4$, $\mu_{4,4} = 11(\frac{\alpha}{2n})^4 + 50(\frac{\alpha}{2n})^3 + 82(\frac{\alpha}{2n})^2 + 56(\frac{\alpha}{2n}) + \frac{49}{3}$.
- *For* $m = 5$, $\mu_{4,5} = 18(\frac{\alpha}{2n})^4 + 108(\frac{\alpha}{2n})^3 + \frac{501}{2}(\frac{\alpha}{2n})^2 + \frac{531}{2}(\frac{\alpha}{2n}) + \frac{5207}{48}$.

Proof. We can easily verify the above results by substituting $m = 2, 3, 4$ and 5 in (3.6). □

Remark 3.14. *The propositions 3.8-3.13 give the presentation of* r^{th} *moments about mean. We conclude that the* r^{th} *moments can be expressed in the form of* r^{th} *degree polynomial. i.e.* $p(x) = a_r x^r + a_{r-1}x^{r-1} + a_{r-2}x^{r-2} + \cdots + a_1 x + a_0$ *where* $x = \frac{\alpha}{2n}$.

Measure of kurtosis: Kurtosis is the measure of whether the data are heavy tailed or light tailed relative to a normal distribution. That is, data set with high kurtosis tends to have heavy tail or outliers. Data set with low kurtosis tend to have light tail, or lack of outliers. Measures of kurtosis can be calculated by [1]. These are given below

$$\beta_{1,m} = \frac{(\mu_{3,m})^2}{(\mu_{2,m})^3}, \tag{3. 10}$$

and

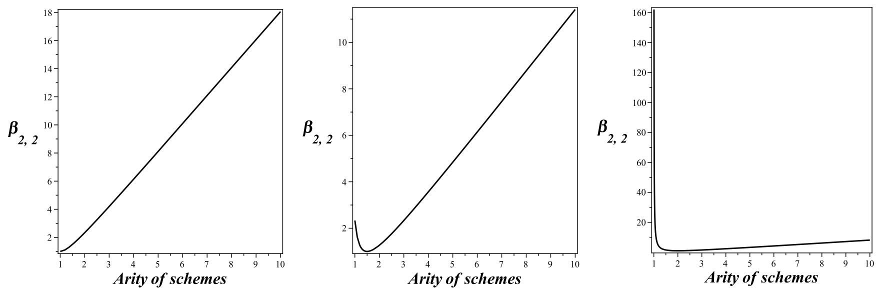
$$\beta_{2,m} = \frac{\mu_{4,m}}{(\mu_{2,m})^2}. \tag{3. 11}$$

The shape of the distribution depends on $\beta_{2,m}$. If $\beta_{2,m} < 3$, $\beta_{2,m} > 3$ and $\beta_{2,m} = 3$ then the distribution is platykurtosis, leptokurtosis and normal distribution.

TABLE 2. *Shows the general formulas of $\beta_{1,m}$ and $\beta_{2,m}$ for $m = 2, 3, 4$ and 5.*

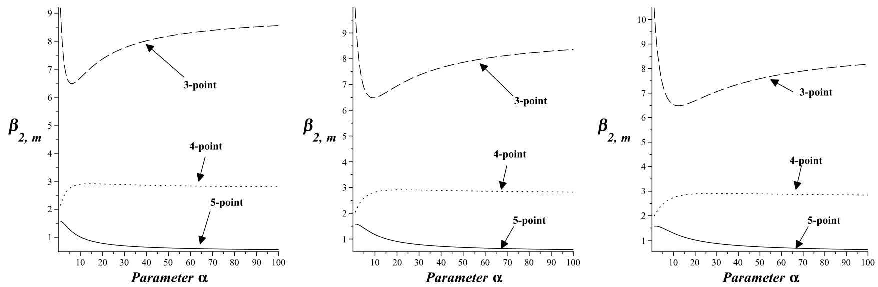
m	$\beta_{1,m} = \frac{(\mu_{3,m})^2}{(\mu_{2,m})^3}$	$\beta_{2,m} = \frac{\mu_{4,m}}{(\mu_{2,m})^2}$
2	$\beta_{1,2} = \frac{4(n-\alpha)^2}{\alpha(2n-\alpha)}$	$\beta_{2,2} = \frac{4n^2-6n\alpha+3\alpha^2}{\alpha(2n-\alpha)}$
3	$\beta_{1,3} = \frac{4(2\alpha^3+3n\alpha^2+14n^2\alpha+3n^3)^2}{(\alpha^2+2n\alpha+n^2)^3}$	$\beta_{2,3} = \frac{54\alpha^2n^2+36n^3\alpha+13n^4+12n\alpha^3+9\alpha^4}{(\alpha^2+2n\alpha+n^2)^2}$
4	$\beta_{1,4} = \frac{(3\alpha^3+18\alpha^2n+40\alpha n^2+32n^3)^2}{2(\alpha^2+4n\alpha+6n^2)^3}$	$\beta_{2,4} = \frac{300\alpha^3n+984\alpha^2n^2+1344\alpha n^3+784n^4+33\alpha^4}{12(\alpha^2+4n\alpha+6n^2)^2}$
5	$\beta_{1,5} = \frac{243(27\alpha^2n+86\alpha n^2+96n^3+3\alpha^3)^2}{(18\alpha^2+54n\alpha+101n^2)^3}$	$\beta_{2,5} = \frac{3(54\alpha^4+648\alpha^3n+3006\alpha^2n^2+6372\alpha n^3+5207n^4)}{(18\alpha^2+54n\alpha+101n^2)^2}$

Using the above distribution theory, we will discuss the kurtosis behavior of the subdivision scheme. In Table 2, we present the general formulas of $\beta_{1,m}$ and $\beta_{2,m}$ for different values of m. The parameters m, n and α are the shape parameters, which characterize the skewness and kurtosis of the subdivision schemes.



(**a**) $\alpha = 1$ (**b**) $\alpha = 1.5$ (**c**) $\alpha = 2$

FIGURE 5. Plots of $\beta_{2,m}$ for $m = 2$ and different values of α and n.



(**a**) Binary (**b**) Ternary (**c**) Quaternary

FIGURE 6. Plots of $\beta_{2,m}$ for different complexity m and arity n.

Remark 3.15. *We have the following concluding remarks from Figure* 5 *and Figure* 6

- *From Figure* 5*, we see that the kurtosis behavior of 2-point* n*-ary subdivision schemes rapidly changes from platykurtosis to leptokurtosis.*
- *From Figure* 6*, 3-point,* n*-ary subdivision schemes are leptokurtosis, 4-point,* n*-ary subdivision schemes are approximately normal and 5-point,* n*-ary subdivision schemes are platykurtosis.*

In general, for all $m \geq 5$*,* $\alpha \geq 1$ *and* $n \geq 2$ *the subdivision scheme* (2.1) *are platykurtosis.*

3.16. **Alternative method for calculating the mean of subdivision scheme.** Here, we present the alternative method for computing the mean of the subdivision scheme.

Theorem 3.17. *If* $b(z)$ *is the Laurent polynomial of the difference scheme of* 2*-point,* n*-ary subdivision schemes then for* $z = \frac{2n-n(n+2)+\alpha}{n^2}$*, we get the mean of* 2*-point,* n*-ary subdivision schemes.*

Proof. This result can be proved by induction on n. For $m = n = 2$ in (2. 2), we get the mask of 2-point binary scheme $a_2^2 = \frac{1}{4}[1, 3, 3, 1]$. The Laurent polynomial of a_2^2 is defined as

$$a(z) = \frac{1}{4}\left(1 + 3z^1 + 3z^2 + z^3\right) = \left(\frac{1+z}{4}\right)^2 b(z), \quad \text{where} \quad b(z) = 1 + z.$$

By substituting $z = \frac{-4+\alpha}{4}$ in the Laurent polynomial $b(z)$ of the difference scheme, we get

$$b\left(\frac{-4+\alpha}{4}\right) = \frac{\alpha}{4} = \frac{nm - (2n-\alpha)}{2n}, \quad \text{for} \quad m = n = 2.$$

Which is the mean of 2-point binary scheme.
Now by substituting $m = 2$ and $n = 3$ in (2.2), the Laurent polynomial of the 2-point ternary scheme is

$$a(z) = \left(\frac{1+z+z^2}{3}\right)^2 b(z), \quad \text{where} \quad b(z) = \frac{3}{2}(1+z).$$

If we substitute $z = \frac{-9+\alpha}{9}$ in $b(z)$, we get

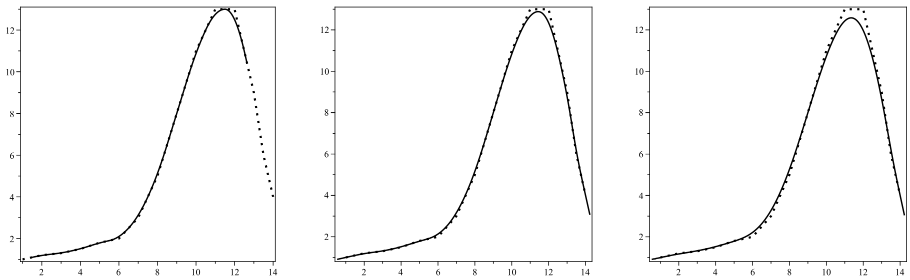
$$b\left(\frac{-9+\alpha}{9}\right) = \frac{\alpha}{6} = \frac{nm - (2n-\alpha)}{2n}, \quad \text{for} \quad m = 2 \quad \text{and} \quad n = 3.$$

Which is the mean of 2-point ternary scheme.
Hence in general by substituting $z = \frac{2n-n(n+2)+\alpha}{n^2}$ in the Laurent polynomial of the difference schemes of 2-point, n-ary subdivision schemes, we get $\mu_{1,m} = \frac{nm-(2n-\alpha)}{2n}$. This completes the proof. □

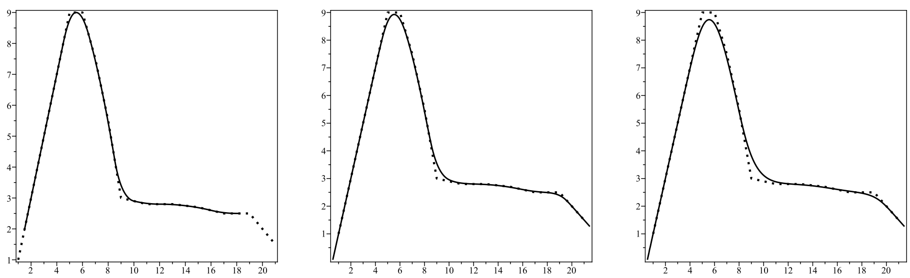
4. NUMERICAL EXPERIMENTS

In this section, we present the numerical experiments of the family of m-point, n-ary subdivision schemes. We apply these schemes on different types of data like negatively skewed, positively skewed and symmetrical data etc. First, we apply the 2-point, 3-point and 5-point binary schemes on negatively (left) skewed data. From the first row of Figure 7, we see that the shapes produced by these schemes also preserve the negatively skewed nature of data. Similarly, the second row of Figure 7 shows the response of binary schemes on the positive (right) skewed data. In case of symmetrical data, the shapes produced by these schemes are symmetrical. It is shown in Figure 3: (a), (b), (c).

In Figure 8, we apply the binary schemes of different complexity on the data obtained from discontinuous function. The Gibbs phenomenon has not been seen in the discontinuity zones of the data. This support the arguments presented in [10].



(**a**) 2-point (**b**) 3-point (**c**) 5-point



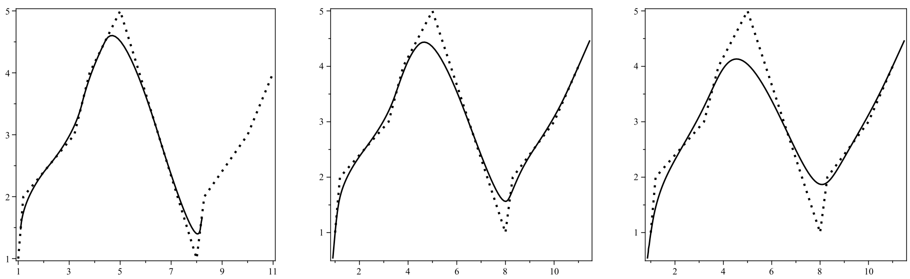
(**d**) 2-point (**e**) 3-point (**f**) 5-point

FIGURE 7. Numerical experiments of binary schemes for different values of m on negative and positive skewed data.

5. CONCLUSIONS

In this paper, statistical analysis of the subdivision schemes has been presented using the theory of probability distribution. We discussed the important properties such as mean, variance, moments about the origin, moments about the mean, measures of skewness and measures of kurtosis of the subdivision schemes. We have concluded the following in general:

- The mean of m-point binary subdivision schemes increases with the increase of m while in general the mean of m-point, n-ary ($n \geq 3$) subdivision schemes decreases.



(a) 2-point binary (b) 3-point binary (c) 5-point binary

FIGURE 8. Plots of binary schemes for Gibbs phenomenon.

- In general, we conclude that as the arity and complexity of the schemes increases the variance of subdivision schemes increases.
- The 2-point, n-ary schemes produce symmetrical shapes when data is symmetrical for $n = \alpha$. These schemes produce positively skewed shapes when date is positively skewed in nature for different values of n and α. But in general the m-point $(m \geq 3)$, n-ary schemes produce negatively skewed shapes when data is negatively skewed.
- The 2-point, n-ary schemes produce the shapes which change from platykurtosis to leptokurtosis shapes for different values of α. All the 3-point, n-ary schemes produce the leptokurtosis shapes, 4-point, n-ary schemes produce normal shapes and 5-point, n-ary schemes produce the platykurtosis shapes.

This work is first attempt to link between univariate subdivision schemes and univariate probability distribution. This paper will open the door for new research directions. Some possible directions are:

- What are the relations between continuous probability distribution, B-splines and subdivision schemes?
- How can we relate bivariate subdivision schemes with multivariate probability distribution?

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