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A Modernistic Approach to Handle Time Fractional Partial Differential Equations by Merging Successive Differentiation Method and Fractional Wave Variable Transformation

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Abstract. In this work, four well known time Fractional Partial Differential Equations (FPDEs) namely, time Fractional Fornberg-Whitham Equation (FFWE), time Fractional KdV Equation (FKdVE), time Fractional Convection-Diffusion Equation (FCDE) and time Fractional BBM-Burger Equation (FBBMBE) are solved numerically through Fractional Wave Variable Transformation (FWVT) and Successive Differentiation Method (SDM). By using the FWVT, $\zeta = \lambda x - \dfrac{vt^{\alpha}}{\Gamma(\alpha+1)}$, these FPDEs are converted into an Ordinary Differential Equation (ODE). Then SDM is applied on thus formed ODE to produce their Taylor series. A comparison of obtained numerical series at $\alpha = 1$ with the exact solution is presented to prove the accuracy of this technique as well as graphical illustrations for different values of α. SDM along with fractional wave variable transformation proves to be an adequate and accurate method for obtaining numerical solutions of FPDEs.

AMS Subject Classification Codes: 35-XX; 35R11; 41A58; 35C07; 35C10
Key Words: Successive Differentiation Method, Fractional Partial Differential Equation, Taylor Series, Series Solution, Fractional Wave Variable.

1. Introduction

Physical phenomena are best described by Partial Differential Equations (PDEs), which were first developed in the 18^{th} century for describing heat and wave phenomenon by Fourier theory [10]. Since then, PDEs have found applications almost in every field of

knowledge: music, chemistry, fluid dynamics [13], quantum mechanics, classical mechanics, biology [8], pharmacology, electrostatics, electrodynamics, economics [33] and many more. PDEs are described in one, two or three dimensions depending on independent variables. (For further detailed study on partial differential equations please see [30]).
Since PDEs describe the complex situations mathematically, it is quite tricky to find the exact solution of every PDE; therefore, researchers have invented various numerical methods to analyze their solutions numerically. Some notable numerical methods that are used in the current era are: Higher Order Compact Finite Difference Method, Restrictive Taylor Series Method [32], New Iterative Method, Residual Power Series Method, Homotopy Perturbation Method (HPM) [1], Optimal q-Homotopy Analysis Transform Method (Oq-HATM), Homotopy Analysis Method (HAM), Differential Transform Method (DTM) [37], Adomian Decomposition Method (ADM), Perturbation Iteration Algorithm (PIA), Successive Differentiation Method (SDM), Variational Iteration Method (VIM), Least Squares Finite Element Method, Haar Wavelet Method, Semi Implicit Finite Difference Scheme, Matrix Free Modified Extended Backward Differential Formula (MF-MEBDF) [18], combination of Method Of Lines (MOL), Chebyshev & Spectral Fourier Methods and the Cubic B-Spline Collocation Method.
Nonlinear Fornberg-Whitham equation is solved numerically to analyze its behaviour by Residual Power Series Method and results have been compared with the exact solution [25]. Murat et al. [29] solved combined KdV-mKdV numerically by Cubic B-Spline Collocation Method. A one dimensional Telegraph equation is solved numerically by using Homotopy Analysis Method and results obtained are approximately near to the exact solution [3]. Al-Badrani et al. [2] used modified Adomian Decomposition Method for solving Telegraph equation numerically. Navier Stokes equation is solved by using higher order Compact Finite difference Method [35]. Similarly, Fishers equation has been solved numerically by two methods such as Least Squares Finite Element Method [7] & Haar Wavelet method [12]. A new technique known as SDM with wave variable transformation was developed recently for finding numerical solutions of highly nonlinear PDEs. In this technique PDEs were converted into ODEs by using a wave variable transformation and then by successively differentiating a numerical solution was obtained by the Taylor series expansion [22]. SDM was initially developed for finding numerical solutions of ordinary differential equation such as Bratu type equations by Wazwaz [36]. Khalid et.al applied SDM to obtain easy and accurate numerical solution of Lane-Emden equation [23].
Researchers and mathematicians developed fractional PDEs for the generalization of interesting phenomena or to see beyond and between the first and second order partial derivatives. Most of the numerical methods mentioned above are capable or are modified for solving time fractional PDEs such as nonlinear FFWE is solved numerically by Oq-HATM [11]. FFWE is also solved by New Iterative Method and modified Homotopy Perturbation Method [19]. FCDE is solved numerically by Variational Iteration Method [26]. Another numerical method proved to be accurate and easier in calculations was Perturbation-Iteration Algorithm for solving time fractional partial differential equation [21].
In this research paper, SDM is utilized to find the numerical solutions of four famous time fractional partial differential equations by incorporating fractional wave variable transformation. Equations that have been solved numerically by SDM are FFWE, FKdVE, FCDE and FBBMBE. All these time fractional partial differential equations will first be converted

into ordinary differential equations by using the fractional wave variable transformation. Then SDM will be applied to obtain their numerical solutions.
This paper consists of the following sections: Section 2 is based on the basic concepts of fractional calculus that will be used in this work. Section 3 is the mathematical formulation of this numerical technique along with fractional wave variable transformation. Section 4 describes the application of this mathematical formulation on four different time fractional PDEs which have application in different fields. Section 5 discusses the conclusion of this work.

2. Preliminary Concepts of Fractional Calculus

Some basic definitions of fractional calculus used in this paper for solving FPDEs are given below with brief description:

Definition 2.1. *The most commonly used definition for fractional derivative is Caputo sense derivative of fractional order α given as*

$$D_t^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}u^{(n)}(\tau)\,d\tau \tag{2.1}$$

where $n-1<\alpha\leq n,\ n\in N,\ t>0$.
For $\alpha=1$ Caputo sense derivative becomes

$$D^{\alpha}u(t) = \frac{du(t)}{dt} \tag{2.2}$$

2.2. **Properties of Caputo Fractional Derivative.** Some properties of Caputo derivative used in this work are

(a) $D_t^{\alpha}t^{\gamma} = \dfrac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}t^{\gamma-\alpha}\ \ ;\quad \gamma>0$
(b) $D_t^{\alpha}\Big(cu(t)\Big) = cD_t^{\alpha}u(t)$, where c is constant.
(c) $D_t^{\alpha}\Big(au(t)+bv(t)\Big) = aD_t^{\alpha}u(t)+bD_t^{\alpha}v(t)$, where a and b are constant.
(d) $D_t^{\alpha}c=0$.

Modified Fractional Chain Rule: Since u is a function of (x,t) and the only fractional derivative required in this work is with respect to t. So the general form of fractional chain rule in [16] is written as $\dfrac{\partial^{\alpha}u}{\partial t^{\alpha}} = \sigma_t'\dfrac{\partial u}{\partial s}\dfrac{\partial^{\alpha}s}{\partial t^{\alpha}}$. Also the authors in [16] claimed that this is only possible if $\sigma_t' = \dfrac{\Gamma(m\alpha+1)}{m\Gamma(1-\alpha+m\alpha)}$. According to this work it can be calculated as $\dfrac{\partial^{\alpha}u}{\partial t^{\alpha}} = \sigma_t'\dfrac{\partial u}{\partial \zeta}\dfrac{\partial^{\alpha}\zeta}{\partial t^{\alpha}}$ where $\sigma_t' = \dfrac{\Gamma(m\alpha+1)}{m\Gamma(1-\alpha+m\alpha)} = C$, where it could be any constant number except **1** [4,6]. For details please see [16]

3. Mathematical Formulation

Consider a nonlinear time fractional PDE of general form as

$$F(u, D_t^{\alpha}u, u_x, u_{xx}, u_{xxx}, \cdots, u_{xt}, \cdots) = 0 \tag{3.3}$$

Let the fractional wave variable transformation be

$$\zeta = \lambda x - \frac{vt^{\alpha}}{\Gamma(\alpha+1)} \tag{3.4}$$

Where λ and v are the constants to be evaluated later that satisfy the Eq.(3.4). This equation transforms the time-fractional PDE into a simple ODE and becomes, $u(x,t) = U(\zeta)$. By using Eq.(3.4), following derivatives can be obtained, $u_x(x,t) = \lambda U'(\zeta)$, $u_{xx}(x,t) = \lambda^2 U''(\zeta)$, $D_t^{\alpha} u(x,t) = -vCU'(\zeta)$ since $D_t^{\alpha}\zeta = -v$ etc (For details see [16]). Then Eq.(3.3) becomes

$$F\Big(U(\zeta), -vCU'(\zeta), \lambda U^{'}(\zeta), \lambda^2 U^{''}(\zeta), \lambda^3 U^{'''}(\zeta), \cdots, \frac{-\lambda v t^{\alpha-1}\alpha U^{''}(\zeta)}{\Gamma(\alpha+1)}, \cdots\Big) = 0 \tag{3.5}$$

where $C = \dfrac{\Gamma(m\alpha+1)}{m\Gamma(1-\alpha+m\alpha)}$. After successfully differentiating Eq.(3.5) many times with respect to ζ, $U^{'''}(\zeta)$, $U^{iv}(\zeta)$, $U^{v}(\zeta), \cdots$ are derived. By taking $\zeta = 0$ in these derivatives, Taylor series can be obtained for $U(\zeta)$ as

$$u(x,t) = U(\zeta) = \sum_{n=0}^{\infty} \frac{U^{(n)}(0)}{n!}\zeta^n \tag{3.6}$$

Eq.(3.4) is then substituted in Eq.(3.6) and becomes

$$u(x,t) = U(\zeta) = \sum_{n=0}^{\infty} \frac{U^{(n)}(0)}{n!}(\lambda x - \frac{vt^{\alpha}}{\Gamma(\alpha+1)})^n \tag{3.7}$$

Find the values of constants λ and v and hence the final numerical solution is obtained by putting the value of these constants in Eq.(3.7).

4. Numerical Examples

Four widely known time fractional partial differential equations have been solved by Successive Differentiation Method. Each of them has its own importance in the respective fields. Since it's not easy to calculate their solutions analytically, in several cases these particular equations are solved numerically.

4.1. **Example 1.**

Let's consider the first example to be FFWE [11], with initial condition $u(x,0) = e^{\frac{-x}{2}}$ written mathematically as

$$D_t^{\alpha} u(x,t) - u_{xxt}(x,t) + u_x(x,t) = u(x,t)\Big(u_{xxx}(x,t) - u_x(x,t)\Big) + 3u_x(x,t)u_{xx}(x,t). \tag{4.8}$$

$u(x,t) = e^{\left|\frac{-x}{2} + \frac{2t}{3}\right|}$ is the exact solution of FFWE. Using fractional wave variable transformation in Eq.(3.4), Eq.(4.8) becomes

$$(\lambda - vC)U^{'} + \frac{\lambda^2 v\alpha t^{\alpha-1}U^{'''}(\zeta)}{\Gamma(\alpha+1)} = \lambda^3 D\Big(-\frac{U^{'}(\zeta)^2}{2} + U(\zeta)U^{''}(\zeta)\Big) -$$

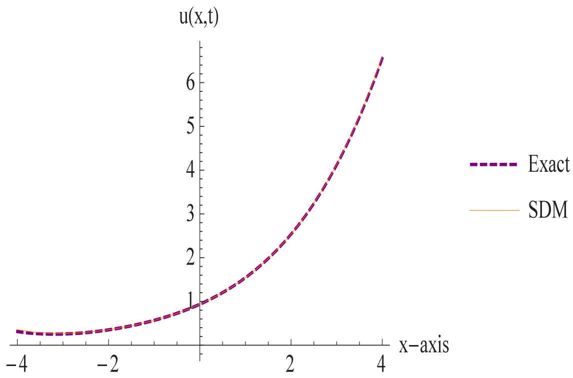

FIGURE 1. Comparison between exact solution and numerical series solution by SDM at $\alpha = 1$, $C = -8290525091498482$ and $t = 0.1$ of Time Fractional Fornberg-Whitham equation.

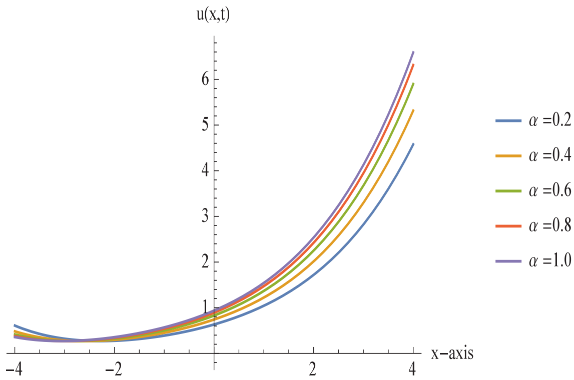

FIGURE 2. Graphical representation of Time Fractional Fornberg-Whitham equation for $\alpha = 0.2, 0.4, 0.6, 0.8, 1.0$ and $C = -8290525091498482$ at $t = 0.1$ and $x \in [-4, 4]$.

$$\lambda D\Big(\frac{U(\zeta)^2}{2}\Big) + 3\lambda^3 D\Big(\frac{U^{'}(\zeta)^2}{2}\Big) \tag{4.9}$$

by integrating Eq.(4.9)

$$(\lambda - vC)U(\zeta) + \frac{\lambda^2 v\alpha t^{\alpha-1}U^{''}(\zeta)}{\Gamma(\alpha+1)} = \lambda^3 U^{'}(\zeta)^2 + \lambda^3 U(\zeta)U^{''}(\zeta) - \frac{\lambda}{2}U(\zeta)^2 \tag{4.10}$$

Rearrange Eq.(4.10) to get

$$U^{''}(\zeta) = -\frac{t\Gamma(\alpha+1)\left(2Cvu(\zeta) + 2\lambda^3 u'(\zeta)^2 - \lambda u(\zeta)^2 - 2\lambda u(\zeta)\right)}{T} \tag{4.11}$$

where $T = 2\lambda^2\left(\lambda t\Gamma(\alpha+1)u(\zeta) - \alpha v t^{\alpha}\right)$. By successively differentiating Eq.(4.11) we get

$$
\begin{aligned}
U^{'''}(\zeta) = \frac{t^2\Gamma(\alpha+1)^2 u'(\zeta)\left(2Cvu(\zeta)+2\lambda^3 u'(\zeta)^2-\lambda u(\zeta)^2-2\lambda u(\zeta)\right)}{\lambda T^2} - \\
\frac{t\Gamma(\alpha+1)\left(2Cvu'(\zeta)-2\lambda u(\zeta)u'(\zeta)-2\lambda u'(\zeta)+4\lambda^3 u'(\zeta)u''(\zeta)\right)}{T} \\
. \\
. \\
.
\end{aligned}
\tag{4.12}
$$

Substituting $\zeta = 0$ in Eq.(4.11), Eq.(4.12). The only initial condition $u(x,0) = e^{\frac{-x}{2}}$ transformed by $\zeta = 0$ iff $x = 0$ and $t = 0$, hence $U(0) = u(0,0) = 1$. Since Eq.(4.11) is the second order derivative, so $U^{'}(0) = a$ is assumed, a will be evaluated later. The following is obtained

$$
\begin{aligned}
U(0) &= 1 \\
U^{'}(0) &= a \\
U^{''}(0) &= -\frac{t\Gamma(\alpha+1)\left(2a^2\lambda^3+2Cv-3\lambda\right)}{T_0} \\
U^{'''}(0) &= \frac{at\Gamma(\alpha+1)\left(\lambda t\Gamma(\alpha+1)\left(6a^2\lambda^3+4Cv-5\lambda\right)+2\alpha v t^{\alpha}(Cv-2\lambda)\right)}{-T_0} \\
&. \\
&. \\
&.
\end{aligned}
\tag{4.13}
$$

where $T_0 = 2\lambda^2\Big(\alpha v t^{\alpha} - \lambda t\Gamma(\alpha+1)\Big)^2$. More successive derivatives can be obtained on similar pattern. Substituting all these values in Eq.(3.7), yields

$$
U(\zeta) = 1 + a(\lambda x - v\frac{t^{\alpha}}{\Gamma(\alpha+1)}) - \frac{t(2a^2\lambda^3+2Cv-3\lambda)\Big(vt^{\alpha}-\lambda x\Gamma(\alpha+1)\Big)^2}{4\lambda^2\Gamma(\alpha+1)\Big(\lambda t\Gamma(\alpha+1)-\alpha v t^{\alpha}\Big)} + \cdots
\tag{4.14}
$$

To find the value of unknown's λ, v and a, assuming $\alpha = 1$ and $t = 0$ in Eq.(4.14):

$$
U(\zeta) = 1 + ax\lambda - \frac{x^2(2a^2\lambda^3+2Cv-3\lambda)}{4(\lambda-v)} + \cdots
\tag{4.15}
$$

In order to compare with initial condition, consider the following series

$$
e^{\frac{-x}{2}} = 1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} + ...
\tag{4.16}
$$

By comparing Eq.(4.15) and Eq.(4.16), we get $\lambda = -\dfrac{1}{2a}$ and $v = -\dfrac{2}{a(4C-1)}$, therefore Eq.(3.4) becomes

$$\zeta = -\frac{x}{2a} + \frac{2t^{\alpha}}{a(4c-1)\Gamma(\alpha+1)} \tag{4.17}$$

Hence by substituting Eq.(4.17) in Eq.(4.14) the series solution for FFWE by SDM is given as

$$\begin{aligned} U(\zeta) =& 1 - \frac{x}{2} + \frac{2t^{\alpha}}{(4C-1)\Gamma(\alpha+1)} + \frac{(4C-5)xt^{\alpha+1}}{(4C-1)\Big(4\alpha t^{\alpha} + (t-4Ct)\Gamma(\alpha+1)\Big)} + \\ & \frac{\Big(16(4C-5)t^{2\alpha} - 5x^2\Gamma(\alpha+1)^2\Big)t}{8(1-4C)^2\Gamma(\alpha+1)\Big((4C-1)t\Gamma(\alpha+1) - 4\alpha t^{\alpha}\Big)} + \\ & \frac{C(11-28C+16C^2)x^2t\Gamma(\alpha+1)}{2(1-4C)^2\Big(-4\alpha t^{\alpha} + (4C-1)t\Gamma(\alpha+1)\Big)} - \\ & \frac{(7+4C)xt\Gamma(2\alpha+2)}{(4C-1)\Big(4\alpha t^{\alpha} - (4C-1)t\Gamma(\alpha+1)\Big)^2} - \\ & \frac{4t^{3\alpha+1}\Big(32(2C-1)\alpha t^{\alpha} + (7-24C-16C^2)t\Gamma(\alpha+1)\Big)}{3(4C-1)^3\Gamma(\alpha+1)^2\Big(4\alpha t^{\alpha} + (1-4C)t\Gamma(\alpha+1)\Big)^2} + \\ & \frac{32(2C-1)\alpha xt\Gamma(3\alpha+1)}{(1-4C)^2\Gamma(\alpha+1)\Big(4\alpha t^{\alpha} + (1-4C)t\Gamma(\alpha+1)\Big)^2} + \\ & \frac{(4C+7)x^2t^{\alpha+2}\Gamma(\alpha+1)}{4\Big(4\alpha t^{\alpha} + (1-4C)t\Gamma(\alpha+1)\Big)^2} - \\ & \frac{x^3t\Gamma(\alpha+1)\Big(-32(2C-1)\alpha t^{\alpha} + (-7+24C+16C^2)t\Gamma(\alpha+1)\Big)}{48\Big(4\alpha t^{\alpha} + (1-4C)t\Gamma(\alpha+1)\Big)^2} + \cdots \end{aligned} \tag{4.18}$$

By further comparison and solving $a = 0$ is obtained. Now to find the most suitable value of C i.e. -8290525091498482 in this case is obtained by parameter optimization technique. Reader can utilize the techniques they are familiar with for parameter $C's$ evaluation such as Bayesian Minimization etc. Similarly this C can be evaluated for different values of α. Fig.1 shows the comparison between exact solution and series solution of time Fractional Fornberg Whitham equation by SDM at $\alpha = 1$ by taking $t = 0.1$, $C = -8290525091498482$ and $x \in [-4, 4]$, which clearly shows the accuracy of SDM. Fig.2 shows the behavior of Eq.(4.18) for different values of α by taking $t = 0.1$, $C = -8290525091498482$ and $x \in [-4, 4]$.
In Table.1, the series solution acquired through SDM is compared numerically by exact

TABLE 1. Comparison of series solution of time Fractional Fornberg Whitham equation obtained through SDM with exact solution and solution obtained by other numerical methods at $\alpha = 1$, $C = -8290525091498482$ and $t = 1$.

x	Exact	SDM	HPM [31]	FVIM [5]	HPTM [14]
0.0	1.94773404	1.94650205	0.50000000	0.50000000	0.46875000
0.5	1.51689680	1.51595253	0.64201270	0.64201270	0.60188691
1.0	1.18136041	1.18108068	0.82436063	0.82436063	0.77283809
1.5	0.92004441	0.92287768	1.05850001	1.05850001	0.99234375
2.0	0.71653131	0.72993827	1.35914091	1.35914091	1.27419461
2.5	0.55803514	0.59846067	1.74517148	1.74517148	1.63609826
3.0	0.43459820	0.53224665	2.24084454	2.24084454	2.10079175
3.5	0.33846542	0.54270150	2.87730134	2.87730134	2.69747000
4.0	0.26359713	0.64883401	3.69452805	3.69452805	3.46362005

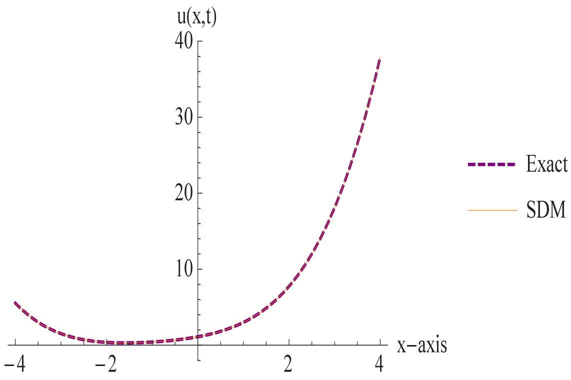

FIGURE 3. Comparison between exact solution and numerical series solution by SDM at $\alpha = 1$ and $C = 0.08610469561565809$ at $t = 0.1$ of Time Fractional Convection-Diffusion equation in $x \in [-4, 4]$.

solution and other contemporary methods at $\alpha = 1$ and $t = 1$. Table.1 clearly shows the results obtained through SDM are more accurate than FVIM, HPM and HPTM.

4.2. **Example 2.** Considered Fractional Convection-Diffusion equation [13] is

$$D_t^{\alpha} u(x,t) = u_{xx}(x,t) - u_x(x,t) + u(x,t)u_{xx}(x,t) - u(x,t)^2 + u(x,t) \tag{4.19}$$

with initial condition and boundary condition as $u(0,t) = e^t$, $u(1,t) = e^{t+1}$ and $u(x,0) = e^x$. The exact solution of Eq.(4.19) is $u(x,t) = e^{x+t}$. Substituting fractional wave variable transformation of Eq.(3.4) in Eq.(4.19) becomes

$$(\lambda - vC)U^{'}(\zeta) = (1 + U(\zeta))\lambda^2 U^{''}(\zeta) + U(\zeta)(1 - U(\zeta)) \tag{4.20}$$

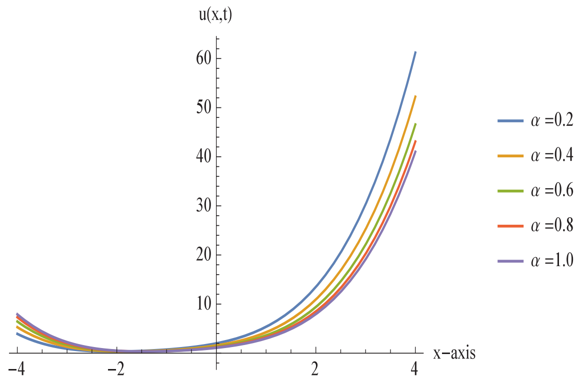

FIGURE 4. Graphical representation of Time Fractional Convection-Diffusion equation for $\alpha = 0.2, 0.4, 0.6, 0.8, 1.0$ at $t = 0.1$ and $x \in [-4, 4]$.

Rearrange Eq.(4.20), to get

$$U^{''}(\zeta) = \frac{(\lambda - Cv)U'(\zeta) + U(\zeta)(U(\zeta) - 1)}{\lambda^2 T} \tag{4.21}$$

where $T = U(\zeta) + 1$. Successively differentiating Eq.(4.21), it becomes

$$U^{'''}(\zeta) = \frac{T_1 U''(\zeta) + (Cv - \lambda)U'(\zeta)^2 + T_2 U'(\zeta)}{\lambda^2 T^2} \tag{4.22}$$

where $T_1 = -U(\zeta + 1)(Cv - \lambda)$ and $T_2 = U(\zeta)^2 + 2U(\zeta) - 1$

$$\begin{aligned} U^{iv}(\zeta) = & \frac{T_2 U''(\zeta) - (U(\zeta) + 1)U^{'''}(\zeta)(Cv - \lambda)}{\lambda^2 T^2} + \\ & \frac{(2\lambda - 2Cv)U'(\zeta)^3 + 3(-T_1 U'(\zeta)U''(\zeta) + 4U'(\zeta)^2}{\lambda^2 T^3} \\ & . \\ & . \\ & . \end{aligned} \tag{4.23}$$

More derivatives can be calculated in the same way. The only initial condition for this problem becomes $U(0) = u(0, 0) = 1$, hence we assume $U^{'}(0) = a$, a will be evaluated later. Now put $\zeta = 0$ in Eq.(4.21), (4.22) & (4.23), they become

$$\begin{aligned} U(0) &= 1 \\ U^{'}(0) &= a \\ U^{''}(0) &= \frac{a(\lambda - Cv)}{2\lambda^2} \\ U^{'''}(0) &= \frac{a\left(C^2v^2 - 2C\lambda v + 3\lambda^2\right)}{4\lambda^4} - \frac{a^2(\lambda - Cv)}{4\lambda^2} \end{aligned}$$

$$U^{iv}(0) = v^2\left(\frac{3aC^2}{8\lambda^5} - \frac{a^2C^2}{2\lambda^4}\right) + v\left(-\frac{a^3C}{4\lambda^2} + \frac{a^2C}{\lambda^3} - \frac{7aC}{8\lambda^4}\right) + \frac{a^3}{4\lambda} - \frac{aC^3v^3}{8\lambda^6} + \frac{5a}{8\lambda^3}$$
$$\vdots \tag{4.24}$$

Substitute values of Eq.(4.24) in Eq.(3.7)

$$U(\zeta) = 1 + a\left(\lambda x - \frac{vt^\alpha}{\Gamma(\alpha+1)}\right) - \frac{a(Cv-\lambda)\left(\lambda x - \frac{vt^\alpha}{\Gamma(\alpha+1)}\right)^2}{4\lambda^2}\cdots \tag{4.25}$$

To find the value of unknown's λ, v and a, taking $\alpha = 1$ and $t = 0$ in Eq.(4.25) it becomes

$$\begin{aligned}U(\zeta) =& 1 + ax\lambda + x^2\Big(\frac{a\lambda}{4} - \frac{aCv}{4}\Big) + x^3\Big(\frac{a^2C\lambda v}{24} - \frac{a^2\lambda^2}{24} + \frac{aC^2v^2}{24\lambda} - \frac{aCv}{12} + \frac{a\lambda}{8}\Big) + \\ & x^4\Big(-\frac{a^3C\lambda^2v}{96} + \frac{a^3\lambda^3}{96} - \frac{a^2C^2v^2}{48} + \frac{a^2C\lambda v}{24} - \frac{aC^3v^3}{192\lambda^2} + \frac{aC^2v^2}{64\lambda} - \\ & \frac{7aCv}{192} + \frac{5a\lambda}{192}\Big) + \cdots\end{aligned} \tag{4.26}$$

In order to compare with initial condition, we consider:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ... \tag{4.27}$$

By comparing Eq.(4.26) and Eq.(4.27) we get $v = -\dfrac{1}{aC}$ and $\lambda = \dfrac{1}{a}$ and by substituting these values Eq.(3.4) becomes

$$\zeta = \frac{x}{a} + \frac{t^\alpha}{aC\Gamma(\alpha+1)} \tag{4.28}$$

Putting Eq.(4.28) in Eq.(4.25) and by simplifying the series a will get vanished during calculations, therefore the series solution for FCDE by SDM is obtained as

$$\begin{aligned}u(x,t) =& 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{t^{4\alpha}}{24C^4\Gamma(\alpha+1)^4} + \frac{t^{3\alpha}}{6C^3\Gamma(\alpha+1)^3} + \\ & \frac{xt^{3\alpha}}{6C^3\Gamma(\alpha+1)^3} + \frac{t^{2\alpha}}{2C^2\Gamma(\alpha+1)^2} + \frac{xt^{2\alpha}}{2C^2\Gamma(\alpha+1)^2} + \frac{x^2t^{2\alpha}}{4C^2\Gamma(\alpha+1)^2} + \\ & \frac{t^\alpha}{C\Gamma(\alpha+1)} + \frac{xt^\alpha}{C\Gamma(\alpha+1)} + \frac{x^2t^\alpha}{2C\Gamma(\alpha+1)} + \frac{x^3t^\alpha}{6C\Gamma(\alpha+1)} + \cdots\end{aligned} \tag{4.29}$$

Now to find the most suitable value of C i.e. 0.08610469561565809 in this case is obtained by parameter optimization technique. Reader can utilize the techniques they are familiar with for parameter $C's$ evaluation such as Bayesian Minimization etc. Similarly this C can be evaluated for different values of α.

Fig.3 shows the comparison between exact solution and series solution of FCDE by SDM at $\alpha = 1$ and $C = 0.08610469561565809$ by taking $t = 0.1$ and $x \in [-4, 4]$, which clearly shows the accuracy of SDM. Fig.4 shows the behavior of Eq.(4.29) for different values of α by taking $C = 0.08610469561565809$, $t = 0.1$ and $x \in [-4, 4]$. In Table.2, the series solution acquired through SDM is compared numerically by exact solution and

TABLE 2. Comparison of series solution of fractional Convection-Diffusion equation obtained through SDM with exact solution and solution obtained by other numerical methods at $\alpha = 1$, $C = 0.08610469561565809$ and $t = 1$.

x	Exact	SDM	HATM [15]	HPSTM [15]	FVIM [24]
0.0	2.71828183	2.70833333	2.66666667	2.70833333	2.70833333
0.1	3.00416602	2.99317101	2.94712245	2.99317124	2.99317124
0.2	3.32011692	3.30795833	3.25707402	3.30796580	3.30796580
0.3	3.66929667	3.65580990	3.59962349	3.65586760	3.65586760
0.4	4.05519997	4.04011111	3.97819919	4.04035856	4.04035856
0.5	4.48168907	4.46451823	4.39659006	4.46528677	4.46528677
0.6	4.95303242	4.93295833	4.85898347	4.93490508	4.93490508
0.7	5.47394739	5.44962934	5.37000722	5.45391358	5.45391358
0.8	6.04964746	6.01900000	5.93477581	6.02750668	6.02750668
0.9	6.68589444	6.64580990	6.55894163	6.66142509	6.66142509
1.0	7.38905610	7.33506944	7.24875154	7.36201329	7.36201329

other contemporary methods at $\alpha = 1$, $C = 0.08610469561565809$ and $t = 1$. Table.2 clearly shows the results obtained through SDM are more accurate than FVIM, HPSTM and HATM.

4.3. **Example 3.** Consider the time Fractional KdV equation [28]

$$D_t^{\alpha} u(x,t) + 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0 \tag{4.30}$$

with initial condition $u(x,0) = \frac{1}{2} sech^2\Big(\frac{x}{2}\Big)$. The exact solution of Eq.(4.30) is $u(x,t) = \frac{1}{2} sech^2\Big(\frac{x-t}{2}\Big)$. Using fractional wave variable transformation in Eq.(3.4), Eq.(4.30) becomes

$$-vCU^{'}(\zeta) + 6\lambda D\Big(\frac{U(\zeta)^2}{2}\Big) + \lambda^3 U^{'''}(\zeta) = 0 \tag{4.31}$$

by integrating Eq.(4.31), it becomes

$$U^{''}(\zeta) = \frac{1}{\lambda^3}\Big(vCU(\zeta) - 3\lambda(U(\zeta))^2\Big) \tag{4.32}$$

By successively differentiating Eq.(4.32):

$$U^{'''}(\zeta) = \frac{U'(\zeta)(vC - 6\lambda U(\zeta))}{\lambda^3} \tag{4.33}$$

$$\begin{aligned} U^{iv}(\zeta) &= \frac{U(\zeta)\Big(v^2C^2 - 9vC\lambda U(\zeta) + 18(\lambda U(\zeta))^2\Big) - 6\lambda^4 (U^{'}(\zeta))^2}{\lambda^6} \\ &. \\ &. \\ &. \end{aligned} \tag{4.34}$$

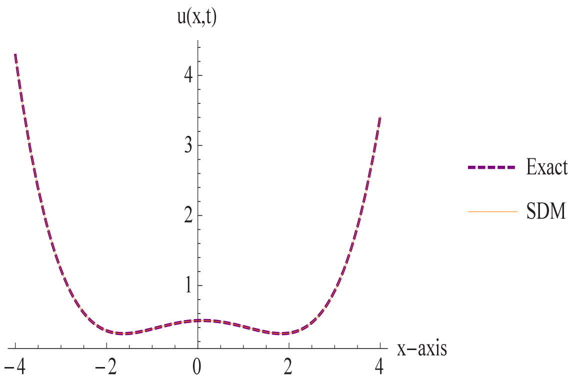

FIGURE 5. Comparison between exact solution and numerical series solution by SDM at $\alpha = 1$, $t = 0.1$ and $C = 0.9859374163793541$ of Time Fractional KdV equation.

On a similar pattern, more successful derivatives can be obtained. Substitute $\zeta = 0$ in Eq.(4.34), it becomes

$$\begin{aligned}
U(0) &= \frac{1}{2}\\
U^{'}(0) &= a\\
U^{''}(0) &= \frac{vC}{2\lambda^3} - \frac{3}{4\lambda^2}\\
U^{'''}(0) &= \frac{avC}{\lambda^3} - \frac{3a}{\lambda^2}\\
U^{iv}(0) &= -\frac{6a^2}{\lambda^2} + \frac{9}{4\lambda^4} + \frac{v^2C^2}{2\lambda^6} - \frac{9vC}{4\lambda^5}\\
&\cdot\\
&\cdot\\
&\cdot
\end{aligned} \tag{4.35}$$

Substituting Eq.(4.35) in Eq.(3.7) yields

$$\begin{aligned}
U(\zeta) =& \frac{1}{2} + a\Big(\lambda x - \frac{vt^\alpha}{\Gamma(\alpha+1)}\Big) + \frac{(2Cv - 3\lambda)\Big(\lambda x - \frac{vt^\alpha}{\Gamma(\alpha+1)}\Big)^2}{8\lambda^3} + \\
& \frac{a(Cv - 3\lambda)\Big(\lambda x - \frac{vt^\alpha}{\Gamma(\alpha+1)}\Big)^3}{6\lambda^3} \cdots
\end{aligned} \tag{4.36}$$

To find value of unknown λ, v and a, consider that $\alpha = 1$ and $t = 0$ in Eq.(4.36):

$$U(\zeta) = \frac{1}{2} + a\lambda x + x^2\Big(\frac{Cv}{4\lambda} - \frac{3}{8}\Big) + x^3\Big(\frac{aCv}{6} - \frac{a\lambda}{2}\Big) + x^4\Big(-\frac{a^2\lambda^2}{4} + \frac{C^2v^2}{48\lambda^2} -$$

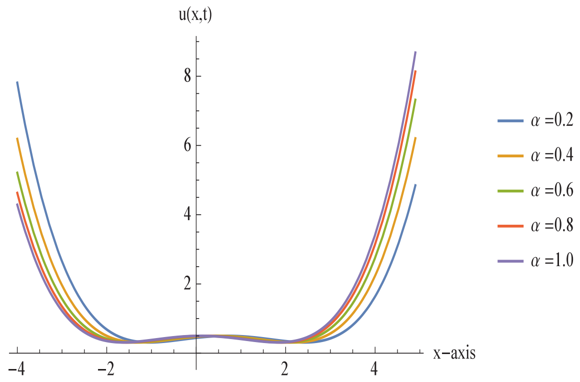

FIGURE 6. Graphical representation of Time Fractional KdV equation for $\alpha = 0.2, 0.4, 0.6, 0.8, 1.0$ at $t = 0.1$ and $x \in [-4, 4]$.

$$\frac{Cv}{32\lambda} - \frac{Cv}{16\lambda} + \frac{3}{32}\Big) \cdots \tag{4.37}$$

In order to compare with initial condition, we consider the series

$$\frac{1}{2}sech^2\Big(\frac{x}{2}\Big) = \frac{1}{2} - \frac{x^2}{8} + \frac{x^4}{48} + ... \tag{4.38}$$

By comparing Eq.(4.37) and Eq.(4.38), we get $v = \frac{\lambda}{C}$ and $a = 0$ therefore by substituting these values Eq.(3.4) becomes

$$\zeta = x\lambda - \frac{\lambda t^\alpha}{C\Gamma(\alpha + 1)} \tag{4.39}$$

Using Eq.(4.39) in Eq.(4.36), and upon simplifying λ will get vanished and the final series solution of FKdVE by SDM is given as

$$\begin{aligned} u(x,t) =& \frac{1}{2} - \frac{x^2}{2} + \frac{x^4}{48} + \frac{t^{4\alpha}}{48C^4\Gamma(\alpha+1)^4} - \frac{xt^{3\alpha}}{12C^3\Gamma(\alpha+1)^3} - \frac{t^{2\alpha}}{8C^2\Gamma(\alpha+1)^2} + \\ & \frac{x^2t^{2\alpha}}{8C^2\Gamma(\alpha+1)^2} + \frac{xt^\alpha}{4C\Gamma(\alpha+1)} - \frac{x^3t^\alpha}{12C\Gamma(\alpha+1)} + \cdots \end{aligned} \tag{4.40}$$

In this case $C = 0.9859374163793541$ is obtained. Fig.5 shows the comparison between exact solution and series solution of FKdVE by SDM at $\alpha = 1$ and $C = 0.9859374163793541$ by taking $t = 0.1$ and $x \in [-4, 4]$, which clearly shows the accuracy of SDM. Fig.6 shows the behavior of Eq.(4.40) for different values of α by taking $t = 0.1$, $C = 0.9859374163793541$ and $x \in [-4, 4]$.

In Table.3, the series solution acquired through SDM is compared numerically by exact solution and other contemporary methods at $C = 0.9859374163793541$, $\alpha = 1$ and $t = 1$. Table.3 clearly shows the results obtained through SDM are more accurate than FVIM, RDTM and ADM.

TABLE 3. Comparison of series solution of fractional KdV equation obtained through SDM with exact solution and solution obtained by other numerical methods at $C = 0.9859374163793541$, $\alpha = 1$ and $t = 1$.

x	Exact	SDM	FVIM [28]	ADM [27]	RDTM [20]
0.0	0.393223866	0.395833333	0.375000000	1.062500000	0.375000000
0.1	0.411000615	0.412033333	0.399914507	1.007299540	0.399914507
0.2	0.427819393	0.427206944	0.424301956	0.919492368	0.424301956
0.3	0.443425747	0.441220833	0.447608106	0.805462604	0.447608106
0.4	0.457568481	0.453950000	0.469273965	0.673827372	0.469273965
0.5	0.470007424	0.465277778	0.488764258	0.534355269	0.488764258
0.6	0.480521491	0.475095833	0.505594819	0.396792393	0.505594819
0.7	0.488916623	0.483304167	0.519356608	0.269791354	0.519356608
0.8	0.495033145	0.489811111	0.529734474	0.160100056	0.529734474
0.9	0.498752080	0.494533333	0.536519415	0.072100640	0.536519415
1.0	0.500000000	0.497395833	0.539613781	0.007714244	0.539613781

4.4. **Example 4.** Considered time fractional BBM-Burger equation [38]

$$D_t^{\alpha} u(x,t) - u_{xxt}(x,t) + u_x(x,t) + \left(\frac{u(x,t)^2}{2}\right)_x = 0 \tag{4.41}$$

with initial condition $u(x,0) = sech^2\left(\frac{x}{4}\right)$. The exact solution of Eq.(4.41) is $u(x,t) = sech^2\left(\frac{x}{4} - \frac{t}{3}\right)$. Using fractional wave variable transformation given in Eq.(3.4), Eq.(4.41) becomes

$$-vU^{'}(\zeta)C + \frac{\alpha v \lambda^2 U^{'''}(\zeta) t^{\alpha-1}}{\Gamma(\alpha+1)} + \lambda U^{'}(\zeta) + \lambda D\left(\frac{U(\zeta)^2}{2}\right) = 0 \tag{4.42}$$

On integrating Eq.(4.42), it becomes

$$U^{''}(\zeta) = -\frac{\Gamma(\alpha+1)t^{1-\alpha}U(\zeta)(-2Cv + 2\lambda + \lambda U(\zeta))}{2\alpha\lambda^2 v} \tag{4.43}$$

By successively differentiating, Eq.(4.43) yields

$$U^{'''}(\zeta) = \frac{\Gamma(\alpha+1)t^{1-\alpha}U'(\zeta)(Cv - \lambda - \lambda U(\zeta))}{\alpha\lambda^2 v} \tag{4.44}$$

$$U^{iv}(\zeta) = -\frac{\Gamma(\alpha+1)t^{1-\alpha}\left(U''(\zeta)(-Cv + \lambda + \lambda U(\zeta)) + \lambda U'(\zeta)^2\right)}{\alpha\lambda^2 v}$$
$$\vdots \tag{4.45}$$

More such derivatives can be calculated in a similar manner. Initial condition by using Eq.(3.4) becomes $U(0) = u(0,0) = 1$, we assume another condition as $U^{'}(0) = a$. Put

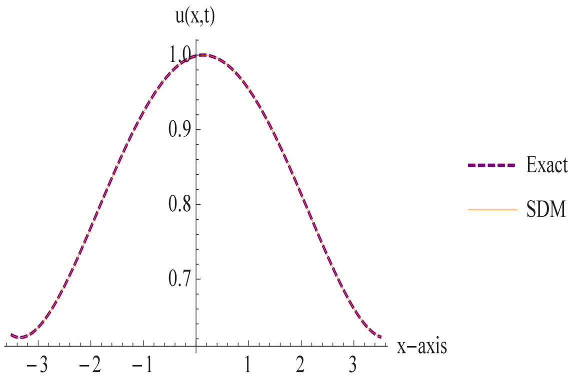

FIGURE 7. Comparison between exact solution and numerical series solution by SDM at $\alpha = 1$, $C = 0.99999999999$ and $t = 0.1$ of Time Fractional BBM-Burger equation.

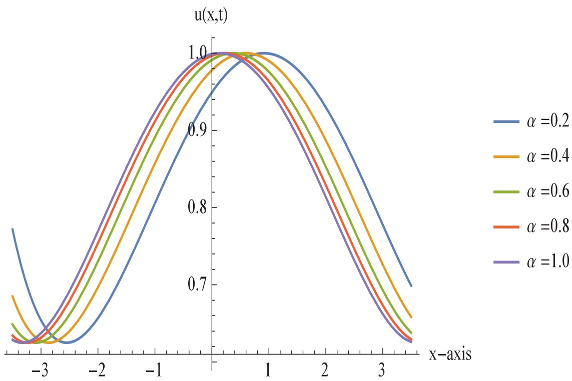

FIGURE 8. Graphical representation of Time Fractional BBM-Burger equation for $\alpha = 0.2, 0.4, 0.6, 0.8, 1.0$ at $t = 0.1$, $C = 0.99999999999$ and $x \in [-3.5, 3.5]$.

$\zeta = 0$ in Eq.(4.43) - Eq.(4.45), they become

$$
\begin{aligned}
U(0) =& 1\\
U^{'}(0) =& a\\
U^{''}(0) =& \frac{\Gamma(\alpha+1)t^{1-\alpha}C}{\alpha\lambda^2} - \frac{\Gamma(\alpha+1)t^{1-\alpha}}{2\alpha\lambda v}\\
U^{'''}(0) =& \frac{a\Gamma(\alpha+1)t^{1-\alpha}C}{\alpha\lambda^2}\\
U^{iv}(0) =& \frac{a^2\Gamma(\alpha+1)t^{1-\alpha}}{\alpha\lambda v} + \frac{\Gamma(\alpha+1)^2t^{2-2\alpha}C^2}{\alpha^2\lambda^4C^2} - \frac{\Gamma(\alpha+1)^2t^{2-2\alpha}C}{2\alpha^2\lambda^3 v}
\end{aligned}
$$

TABLE 4. Comparison of series solution of fractional BBM-Burger equation obtained through SDM with exact solution and solution obtained by other numerical methods at $C = 0.99999999999$, $\alpha = 1$ and $t = 1$.

x	Exact	SDM	FHATM [34]	MRPSM [17]	HAM [9]
0.0	0.896629560	0.897119342	0.750000000	0.500000000	0.750000000
0.1	0.910646768	0.910916494	0.775438735	0.528395369	0.825381485
0.2	0.923830987	0.923875889	0.801624332	0.563307094	0.901167329
0.3	0.936124523	0.935943849	0.828338754	0.604097705	0.976801951
0.4	0.947472404	0.947069454	0.855337838	0.649862148	1.051713440
0.5	0.957822885	0.957204945	0.882356241	0.699485045	1.125323120
0.6	0.967127929	0.966306133	0.909113103	0.751710633	1.197055430
0.7	0.975343665	0.974332808	0.935318229	0.805218015	1.266347990
0.8	0.982430819	0.981249141	0.960678564	0.858694434	1.332661290
0.9	0.988355106	0.987024097	0.98490473	0.910900345	1.395488040
1.0	0.993087580	0.991631837	1.00771741	0.960721759	1.454361550

$$U^{v}(0) = \frac{a\Gamma(\alpha+1)^2 t^{2-2\alpha} C^2}{\alpha^2\lambda^4 C^2} + \frac{3a\Gamma(\alpha+1)^2 t^{2-2\alpha} C}{\alpha^2\lambda^3 v} - \frac{3a\Gamma(\alpha+1)^2 t^{2-2\alpha}}{2\alpha^2\lambda^2 v^2}$$
$$\vdots \tag{4.46}$$

Put Eq.(4.46) in Eq.(3.7) it becomes

$$U(\zeta) = 1 + a\Big(\lambda x - \frac{vt^\alpha}{\Gamma(\alpha+1)}\Big) - \frac{\Gamma(\alpha+1)t^{1-\alpha}(3\lambda - 2Cv)\Big(\lambda x - \frac{vt^\alpha}{\Gamma(\alpha+1)}\Big)^2}{4\alpha\lambda^2 v} + \frac{a\Gamma(\alpha+1)t^{1-\alpha}(Cv - 2\lambda)\Big(\lambda x - \frac{vt^\alpha}{\Gamma(\alpha+1)}\Big)^3}{6\alpha\lambda^2 v} + \cdots \tag{4.47}$$

To find value of unknown λ, v and a, take $\alpha = 1$ and $t = 0$ in Eq.(4.47), it becomes

$$U(\zeta) = 1 + a\lambda x + \frac{a\lambda x^3(Cv - 2\lambda)}{6v} - \frac{x^2(3\lambda - 2Cv)}{4v} + \frac{x^4\Big(-2\lambda^2\left(a^2\lambda v - 3\right) + 2C^2v^2 - 7C\lambda v\Big)}{48v^2} + \cdots \tag{4.48}$$

In order to compare with initial solution, the following series is obtained

$$sech^2\Big(\frac{x}{4}\Big) = 1 - \frac{x^2}{16} + \frac{x^4}{384} + \ldots \tag{4.49}$$

By comparing Eq.(4.48) and Eq.(4.49) we get $a = 0$, $\lambda = \frac{v}{12} + \frac{2vC}{3}$, and v, therefore Eq.(3.4) becomes

$$\zeta = x\left(\frac{2cv}{3} + \frac{v}{12}\right) - \frac{vt^{\alpha}}{\Gamma(\alpha+1)} \tag{4.50}$$

Putting Eq.(4.50) in Eq.(4.47), the final series solution for FBBMBE by SDM is obtained as

$$\begin{aligned} U(\zeta) =& 1 - \frac{9t^2}{(8C+1)^2} + \frac{18(2C+1)t^4}{(8C+1)^4} + \frac{3tx}{2(8C+1)} - \frac{6(2C+1)t^2x}{(8C+1)^3} - \\ & \frac{1}{16} - \frac{3(2C+1)t^2x^2}{4(8C+1)^2} + \frac{(2C+1)tx^3}{24(8C+1)} + \frac{(2C+1)x^4}{1152} + \cdots \end{aligned} \tag{4.51}$$

In this example the most suitable $C = 0.99999999999$ is obtained. Fig.7 shows the comparison between exact solution and series solution of FBBMBE by SDM at $C = 0.99999999999$, $\alpha = 1$ by taking $t = 0.1$ and $x \in [-4, 4]$, which clearly shows the accuracy of SDM. Fig.8 shows the behavior of Eq.(4.51) for different values of α by taking $t = 0.1$, $C = 0.99999999999$ and $x \in [-4, 4]$.

In Table.4, the series solution acquired through SDM is compared numerically by exact solution and other contemporary methods at $\alpha = 1$, $C = 0.99999999999$ and $t = 1$. Table.4 clearly shows the results obtained through SDM are more accurate than FHATM , MRPSM and HAM.

5. Concluding Remarks

This research paper employed the Successive Differentiation Method with a great degree of success on some well known and highly Nonlinear Fractional Partial Differential Equations (NFPDEs). The yielding results evinced the efficiency of the techniques used in this paper.

All eight graphical illustrations of the solved examples shows the effectiveness of this method. The accurate convergence with exact solution can easily be observed among them. The core objective is to show the potency and strong features of SDM over FPDE, in fact through curtailing the order and minimizing the efforts made in time consuming calculations. The substitution of fractional wave variable transformation into SDM is the simpler approach to obtain the numerical solution of almost any type of FPDE. The range of methods like HPM, PIA, DTM, ADM, VIM, HAM, etc involve lengthy and arduous calculations like Adomian Polynomials, Lagrange Multipliers, Perturbation parameters, decomposition of linear or non-linear terms etc, whereas the method presented in this paper is straightforward.

Calculations here are carried out in Mathematica 10.0. The only drawback found is of the choice of the most accurate value of λ, v and C among the various resulted values even complex values can give optimum results in fractional partial differential equations. In this regard no considerable rule has been formulated yet, except for the verification of each value through trial and error technique.

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