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Analysis of Time-Fractional Semi-Analytical Solutions of Strong Interacting Internal Waves in Rotating Ocean

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Abstract.: In this paper, time-fractional Gardner's Ostrovsky equation is considered which represents the shallow water wave phenomena of strong interacting internal Waves with rotational effects. Using the novel perturbation technique, we found the semi-analytical solutions of such obscure phenomena for the rotational parameters introduced in fractional time domain. The Homotopy Perturbation Method is implemented in conjunction with Laplace transformation. Caputo's time fractional derivative has been used to obtain the upcoming solutions on the basis of all previous backgrounds.

AMS (MOS) Subject Classification Codes: 34A08; 35R11
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1. INTRODUCTION

In recent centuries, the fractional calculus has been significantly discussed in engineering and applied sciences. Progress in fractional calculus is reported in different applications in differential equations, plasma physics, signal processing, fluid dynamics, viscoelastic models, biological sciences, and electrochemistry [10, 15, 6, 23, 9, 8].

Undoubtedly, fractional calculus is an efficient mathematical tool to analyze the solution of different problems in mathematics, engineering and sciences. To get more attention in the field and to confirm its effectiveness, this paper is dedicated to recent applications of fractional calculus in engineering sciences [5]. Recently, time-fractional derivatives has been introduced in different nonlinear problems to study nonlinearity effects in the solution [3, 4, 19]. The

There are different conditions when it is necessary to use dual nonlinear terms. In the study of oceanic waves, where internal gravity waves are observed, the only one nonlinear term can not completely model the phenomena of shallow water waves. In the Coastal Ocean Probe Experiment (COPE), it was studied that the internal shallow waves were very strong. COPE was experimented in Oregon Bay in 1995 [2]. Such engineering experiments of shallow water wave phenomenon used for the construction of a differential equation with dual-power law nonlinearity. The equation which illustrates this phenomenon is called Gardner's Equation.

The Gardner-Ostrovsky equation, which is the modified Korteweg de-Vries (KdV) equation with extended rotational effects to elucidate long internal waves of large amplitude [11], where c, α and α_1 represents velocity of dispersionless linear waves, coefficient of quadratic and cubic nonlinearities respectively. Moreover, the coefficients of small and large-scale dispersion are symbolized by β and γ. $\phi(x,t)$ describes a perturbation - from its rest position in isopycnal surface, which is the surface of equal density. In this equation, the dispersion due to nonhydrostaticity produced by the finiteness in depth of the basin and Earth's rotation comes together. The equation involving two nonlinear terms proportional to α and α_1. The first nonlinear term having quadratic nonlinearity which comes traditionally, that is due to hydrodynamic nonlinear system [22], whereas the second term come out either when the first nonlinear term tern to be arbitrarily small (this condition may happen when the pycnocline is appeared close to the half depth of the basin [2,11]). The Gardner's Ostrovsky equation is considered to describe the strong nonlinearity effects produced by large-amplitude waves [16].

Mostly, the analytical solutions to nonlinear fractional differential equations are not available, so the solutions can be obtained by semi-analytical methods to analyze the solutions of the nonlinear dynamical problems [24]. Adomian Decomposition Method (ADM) [21], Variational Iteration Method (VIM)[22], Homotopy Perturbation Method (HPM) [25]. Homotopy Perturbation Method in association with the Laplace Transform Method [3], Homotopy Analysis Method (HAM) [7] and Homotopy Analysis Laplace Transform Method [9,18] and various other methods are used to obtain the solutions to the linear and nonlinear problem.

In this paper, we have solved three examples, Ostrovsky equation, Gardner's equation and Gardner's Ostrovsky equations with time-fractional derivative using Homotopy Perturbation Laplace Transform Method. The method is the combination of classical perturbation method and Laplace transform method. Time fractional model have been considered to understand the solution based on all of its historical states of the solution [14]. The key advantage of the method is (a) use of initial conditions (avoid the boundary conditions without any discretization) (b) Linearization (c) restrictive assumptions to the nonlinear partial and fractional differential equations [3]. Moreover, time-fractional derivative is used to visualize the hidden nonlinear behaviour of the wave while changing the value of the fractional order which can not be obtained by considering the integer order derivative. The method is efficient and reliable for linear equations as well as nonlinear equations of fractional order in Caputo's sense.

2. METHOD DESCRIPTION

To illustrate the methodology of Homotopy Perturbation Laplace Transform Method (HPLTM), we consider the generalized nonlinear differential equation with Caputo's fractional derivative

$$D_t^{n\alpha}\phi(\xi,t) + R\phi(\xi,t) + N\phi(\xi,t) = q(\xi,t); \quad t>0; \quad n-1 < n\alpha \leq n. \tag{2. 1}$$

having the initial condition

$$\phi(\xi,0) = h(\xi).$$

Where R is the linear and N is the general nonlinear operator in ξ and $q(\xi,t)$ is continuous function. The first step towards the solution is applying Laplace transform on Eq. (2. 1), we obtain

$$\mathcal{L}\Big\{D_t^{n\alpha}\phi(\xi,t)\Big\} + \mathcal{L}\Big\{R\phi(\xi,t) + N\phi(\xi,t)\Big\} = \mathcal{L}\Big\{q(\xi,t)\Big\}.$$

Applying Laplace transform of fractional derivative [3], we get

$$\mathcal{L}\Big\{\phi(\xi,t)\Big\} = s^{-1}h(\xi) + s^{-n\alpha}\mathcal{L}\Big\{q(\xi,t)\Big\} - s^{-n\alpha}\mathcal{L}\Big\{R\phi(\xi,t) + N\phi(\xi,t)\Big\}. \tag{2. 2}$$

Taking the inverse Laplace transform on Eq. (2. 2), we find

$$\phi(\xi,t) = I(\xi,t) - \mathcal{L}^{-1}\Big\{s^{-n\alpha}\mathcal{L}\Big\{R\phi(\xi,t) + N\phi(\xi,t)\Big\}\Big\}, \tag{2. 3}$$

where $I(\xi,t)$ is the term obtained from the source term and the given initial conditions. Now implementing the perturbational technique, we can suppose that the solution can be written as a power series in p as given below

$$\phi(\xi,t) = \sum_{n=0}^{\infty} p^n\phi_n(\xi,t) = p^0\phi_0 + p^1\phi_1 + \dots \tag{2. 4}$$

where the parameter $p \in [0,1]$ is consider as a small. Nonlinear term can be written as

$$N\phi(\xi,t) = \sum_{n=0}^{\infty} p^n H_n(\phi) = p^0 H_0(\phi) + p^1 H_1(\phi) + \dots.$$

Here, H_n represents He's polynomials of $\phi_0, \phi_1, \phi_2, ..., \phi_n$ and can be obtained by the formula

$$H_n = \frac{1}{n!}\partial_p^n\left[N\left(\sum_{i=0}^{\infty} p^i\phi_i\right)\right]_{p=0}, n = 0,1,2,... \tag{2. 5}$$

Substituting Eqs. (2. 4) and (2. 5) in Eq. (2. 3), we obtain

$$\sum_{n=0}^{\infty} p^n \phi_n(\xi,t) = I(\xi,t) - \mathcal{L}^{-1}\Big\{ s^{-n\alpha}\mathcal{L}\Big\{ R\sum_{n=0}^{\infty} p^n \phi_n(\xi,t) + \sum_{n=0}^{\infty} p^n H_n(\phi)\Big\}\Big\}.$$

This is combining of the Laplace transformation and Homotopy Perturbation Method by using He's polynomials. Now, equating the coefficient of like powers of p on both sides, the approximate solutions are computed as follows

$$p^0 : \phi_0(\xi,t) = I(\xi,t),$$

$$p^1 : \phi_1(\xi,t) = \mathcal{L}^{-1}\Big\{ s^{-n\alpha}\mathcal{L}\Big\{ R\phi_0(\xi,t) + H_0(\phi)\Big\}\Big\},$$

$$p^2 : \phi_2(\xi,t) = \mathcal{L}^{-1}\Big\{ s^{-n\alpha}\mathcal{L}\Big\{ R\phi_1(\xi,t) + H_1(\phi)\Big\}\Big\},$$

$$p^3 : \phi_3(\xi,t) = \mathcal{L}^{-1}\Big\{ s^{-n\alpha}\mathcal{L}\Big\{ R\phi_2(\xi,t) + H_2(\phi)\Big\}\Big\}.$$

In the similar manner, the other components $\phi_n(\xi,t)$ for all $n > 3$ can be obtained. Lastly, we get the approximate analytical solution $\phi(\xi,t)$ by the truncation

$$\phi(\xi,t) = \lim_{N\to\infty} \sum_{n=1}^{N} \phi_n(\xi,t)$$

3. Time-fractional Ostrovsky Equation

In this section, we consider time-fractional Ostrovsky equation to study the solitary wave solutions with rotational effects.

$$(\phi_t^{\alpha} + \alpha_1\phi\phi_\xi + \beta\phi_{\xi\xi\xi})_\xi = c\phi, \tag{3. 6}$$

subject to the initial condition

$$\phi(\xi,0) = A\,\mathrm{sech}^2\,\xi.$$

It can be written as

$$\phi_t^{\alpha} + \alpha_1\phi\phi_\xi + \beta\phi_{\xi\xi\xi} = c\int \phi d. \tag{3. 7}$$

Laplace transform of Eq. (3. 7) can give the expression of the form

$$\mathcal{L}\Big\{\phi_t^{\alpha} + \alpha_1\phi\phi_\xi + \beta\phi_{\xi\xi\xi}\Big\} = c\mathcal{L}\Big\{\int \phi d\xi\Big\}.$$

Rearranging

$$\mathcal{L}\Big\{\phi_t^{\alpha}\Big\} = -\alpha_1\mathcal{L}\Big\{\phi\phi_\xi\Big\} - \beta\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\} + c\mathcal{L}\Big\{\int \phi d\xi\Big\}.$$

Applying the rule of Laplace transform of fractional derivative, we get

$$s^{\alpha}\phi(\xi,s) - s^{\alpha-1}\phi(\xi,0) = -\alpha_1\mathcal{L}\Big\{\phi\phi_\xi\Big\} - \beta\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\} + c\mathcal{L}\Big\{\int \phi d\xi\Big\}.$$

On simplifying,

$$\phi(\xi,s) = s^{-1}\phi(\xi,0) + s^{-\alpha}\Big\{ -\alpha_1\mathcal{L}\Big\{\phi\phi_\xi\Big\} - \beta\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\} + c\mathcal{L}\Big\{\int \phi d\xi\Big\}\Big\}. \tag{3. 8}$$

By applying Inverse Laplace transform on Eq. (3. 8), that is

$$\phi(\xi,t)=\mathcal{L}^{-1}\Big\{s^{-1}\phi(\xi,0)\Big\}+\mathcal{L}^{-1}\Big\{s^{-\alpha}\Big\{-\alpha_1\mathcal{L}\Big\{\phi\phi_\xi\Big\}-\beta\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\}+c\mathcal{L}\Big\{\int\phi d\xi\Big\}\Big\}\Big\}. \tag{3. 9}$$

On putting the given initial condition in equation (3.4), we have

$$\phi(\xi,t)=A\operatorname{sech}^2\xi+\mathcal{L}^{-1}\Big\{s^{-\alpha}\Big\{R\phi(\xi,t)+N\phi(\xi,t)\Big\}\Big\}. \tag{3. 10}$$

We assume that our solution can be written as power series

$$\phi(\xi,t)=\sum_{n=0}^{\infty}p^n\phi_n(\xi,t)=p^0\phi_0+p^1\phi_1+p^2\phi_2+...$$

Moreover, nonlinear term can be obtained as

$$N\phi(\xi,t)=\sum_{n=0}^{\infty}p^nH_n(\phi)=p^0H_0+p^1H_1+p^2H_2+...$$

Now, using the expression of the He's Polynomial in nonlinearity term and the linear operator in Eq.(3.5), we find the series of approximate solutions

$$\phi_0(\xi,t)=A\operatorname{sech}^2\xi,$$

$$\phi_1(\xi,t)=\frac{t^\alpha}{\Gamma(1+\alpha)}\Big\{A\tanh\xi+2\alpha_1A^2\operatorname{sech}^4\xi\tanh\xi-16\beta A\operatorname{sech}^4\xi\tanh\xi-8\beta A\operatorname{sech}^2\xi\tan^3\xi\Big\},$$

$$\begin{aligned}\phi_2(\xi,t)=\frac{At^{2\alpha}\operatorname{sech}^2\xi}{8\Gamma(1+2\alpha)}&\Big\{16(\alpha_1A(A+14\beta)-8\beta(A+17\beta))\operatorname{sech}^6\xi-96(\alpha_1A(A+22\beta)\\&-10\beta(A+24\beta))\operatorname{sech}^4\xi\tanh^2\xi+\operatorname{sech}^2\xi c^2\cosh^4\xi\log(\cosh\xi)\\&+c(8A-4\alpha_1A+48\beta+3c\log(\cosh\xi))+256(-A+4\alpha_1A-57\beta)\beta\tanh^4\xi\\&+4c(-4\beta+c\log(\cosh\xi))-4c(A+3\beta)+c\log(\cosh\xi)\tanh^2\xi+128\beta^2\tanh^6\xi\Big\}.\end{aligned}$$

Adding the above approximations, that is

$$\begin{aligned}\phi(\xi,t)=\lim_{N\to\infty}\sum_{n=0}^{N}\phi_n(\xi,t)=A\operatorname{sech}^2\xi+\frac{At^{\alpha_1}\tanh\xi}{\Gamma(1+\alpha)}\Big\{c+2(\alpha_1A-8\beta)\operatorname{sech}^4\xi+8\beta\operatorname{sech}^2\xi\tanh^2\xi\Big\}\\+\frac{At^{2\alpha_1}\operatorname{sech}^2\xi}{8\Gamma(1+2\alpha)}\Big\{(16(\alpha_1A(A+14\beta)-8\beta(A+17\beta))\operatorname{sech}^6\xi-96(\alpha_1A(A+22\beta)\\-10\beta(A+24\beta))\operatorname{sech}^4\xi\tanh^2\xi+\operatorname{sech}^2\xi(c^2\cosh4\xi\log(\cosh\xi))\\+c(8A-4\alpha_1A+48\beta+3c\log(\cosh\xi))+256(-A+4\alpha A-57\beta)\beta\tanh^4\xi\\+4(c(-4\beta+c\log[\cosh\xi]))+c(-4(A+3\beta))+c\log[\cosh\xi]\tanh^2\xi+128\beta^2\tanh^6\xi\Big\}.\end{aligned}$$

Figure:1 depicts the numerical solutions of the waves obtained by solving time-fractional Ostrovsky equation for different values of fractional order derivative. The simulation obtained from the values of α presents the nonlinear behaviour of the wave at time $t=0.1,0.5$

and 1 to study the wave profile under the domain 0 to 1. One can easily visualize of the hidden nonlinear effects of the solution in Figure:1 and can be utilize for implementation.

4. Time-fractional Gardner's Equation

Consider the time-fractional Gardner's equation

$$\phi_t^{\alpha} + \alpha_1 \phi \phi_{\xi} + \beta \phi^2 \phi_{\xi} + \gamma \phi_{\xi\xi\xi} = 0, \tag{4. 11}$$

having the initial condition

$$\phi(\xi, 0) = A \operatorname{sech}^2 \xi.$$

Here, taking Laplace Transform on Eq. (4. 11), we get

$$\mathcal{L}\Big\{\phi_t^{\alpha} + \alpha_1 \phi \phi_{\xi} + \beta \phi^2 \phi_{\xi} + \gamma \phi_{\xi\xi\xi}\Big\} = \mathcal{L}\Big\{0\Big\},$$

Rearranging the above equation, we have

$$\mathcal{L}\Big\{\phi_t^{\alpha}\Big\} = -\alpha_1 \mathcal{L}\Big\{\phi\phi_{\xi}\Big\} - \beta\mathcal{L}\Big\{\phi^2\phi_{\xi}\Big\} - \gamma\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\}.$$

Now, implementing the Laplace transform of fractional derivative, we obtain

$$s^{\alpha}\phi(\xi, s) - s^{\alpha-1}\phi(\xi, 0) = -\alpha_1 \mathcal{L}\Big\{\phi\phi_{\xi}\Big\} - \beta\mathcal{L}\Big\{\phi^2\phi_{\xi}\Big\} - \gamma\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\}.$$

On simplifying,

$$\phi(\xi, s) = s^{-1}\phi(\xi, 0) + s^{-\alpha}\Big\{-\alpha_1 \mathcal{L}\Big\{\phi\phi_{\xi}\Big\} - \beta\mathcal{L}\Big\{\phi^2\phi_{\xi}\Big\} - \gamma\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\}\Big\}. \tag{4. 12}$$

Now, we take Laplace inverse transform on Eq. (4. 12), we obtain the expression

$$\phi(\xi, t) = \mathcal{L}^{-1}\Big\{s^{-1}\phi(\xi, 0)\Big\} + \mathcal{L}^{-1}\Big\{s^{-\alpha}\Big\{-\alpha_1 \mathcal{L}\Big\{\phi\phi_{\xi}\Big\} - \beta\mathcal{L}\Big\{\phi^2\phi_{\xi}\Big\} - \gamma\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\}\Big\}\Big\}. \tag{4. 13}$$

By putting the initial condition in Eq. (4. 12), we have

$$\phi(\xi, t) = A \operatorname{sech}^2 \xi + \mathcal{L}^{-1}\Big\{s^{-\alpha}\Big\{R\phi(\xi, t) + N\phi(\xi, t)\Big\}\Big\}. \tag{4. 14}$$

We assume that our solution can be written as power series

$$\phi(\xi, t) = \sum_{n=0}^{\infty} p^n \phi_n(\xi, t) = p^0\phi_0 + p^1\phi_1 + p^2\phi_2 + ...$$

and the nonlinear term can be represented as

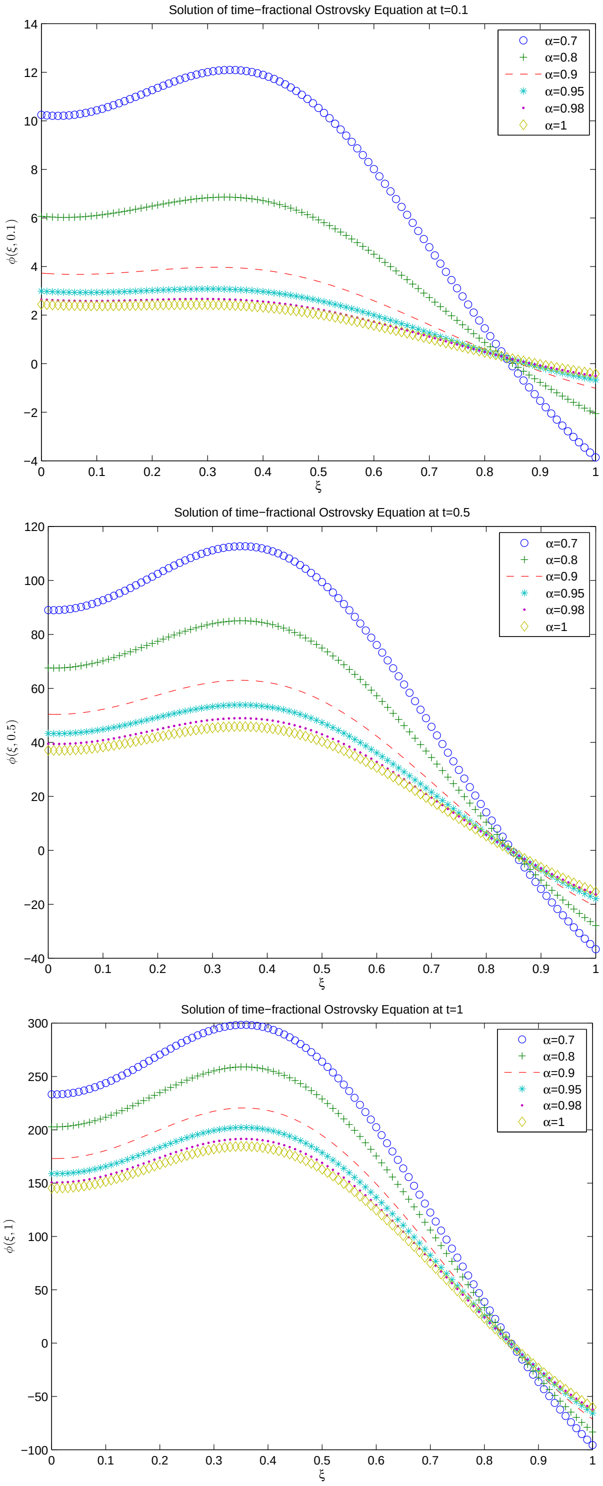
$$N\phi(\xi, t) = \sum_{n=0}^{\infty} p^n H_n(\phi) = p^0 H_0 + p^1 H_1 + p^2 H_2 + ...$$

Now, using the He's Polynomial in nonlinearity term and the linear operator in Eq.(4. 14), we can obtain the solution

$$\phi_0(\xi, t) = A \operatorname{sech}^2 \xi,$$

$$\phi_1(\xi, t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)}\Big\{-2\alpha_1 A^2 sech^4 \xi \tanh \xi - 2A^3\alpha_2 \operatorname{sech}^6 \xi \tanh \xi + \beta(16A \operatorname{sech}^4 \xi \tanh \xi - 8A \operatorname{sech}^2 \xi \tanh^3 \xi)\Big\}$$

FIGURE 1. Numerical Results of Example:1 for $\alpha = 1, 0.98, 0.95, 0.9, 0.8$ and 0.7



$$\phi_2(\xi,t) = \frac{At^{2\alpha}}{8\Gamma(1+2\alpha)}\Big\{ -12\alpha_1^2A^2 + 354\alpha_1 A\beta - 2604\beta^2 - 48\alpha_1 A^3\alpha_2 + 1152A^2\beta\alpha_2 - 96A^4\alpha_2^2 + \cosh 2\xi(-11\alpha_1^2A^2 + 272\alpha_1\beta A - 1806\beta^2 - 16\alpha_1\alpha_2 A^2 + 80A^4\alpha_2^2) + 3\alpha_1^2A^2\cosh 6\xi - 96\alpha_1\beta A\cosh 6\xi + 717\beta^2\cosh 6\xi + 120A^2\beta\alpha_2\cosh 6\xi + 10A\alpha_1\beta\cosh 8\xi - 116\beta^2\cosh 8\xi + \beta^2\cosh 10\xi\Big\}\operatorname{sech}^{12}\xi$$

Adding the above approximations, that is

$$\phi(\xi,t) = \lim_{N\to\infty}\sum_{n=0}^{N}\phi_n(\xi,t) = A\operatorname{sech}^2\xi + \Big\{-2\alpha_1A^2\operatorname{sech}^4\xi\tanh\xi - 2A^3\alpha_2\operatorname{sech}^6\xi\tanh\xi + \beta(16A\operatorname{sech}^4\xi\tanh\xi - 8A\operatorname{sech}^2\xi\tanh^3\xi)\Big\} + \frac{At^{2\alpha}}{8\Gamma(1+2\alpha)}\Big\{-12\alpha_1^2A^2 + 354\alpha_1A\beta - 2604\beta^2 - 48\alpha_1A^3\alpha_2 + 1152A^2\beta\alpha_2 - 96A^4\alpha_2^2 + \cosh 2\xi(-11\alpha_1^2A^2 + 272\alpha_1\beta A - 1806\beta^2 - 16\alpha_1\alpha_2A^2 + 80A^4\alpha_2^2) + 3\alpha_1^2A^2\cosh 6\xi - 96\alpha_1\beta A\cosh 6\xi + 717\beta^2\cosh 6\xi + 120A^2\beta\alpha_2\cosh 6\xi + 10\alpha_1\beta A\cosh 8\xi - 116\beta^2\cosh 8\xi + \beta^2\cosh 10\xi\Big\}\operatorname{sech}^{12}\xi.$$

Figure:2 present the numerical solutions obtained by solving time-fractional Gardner's equation for different values of α. The simulation obtained from the values of α to study the wave profile in $0 < t \leq 1$ and depicts that the nonlinear behaviour of the wave is more visible when putting the fractional values of the order of the derivative at time $t = 0.1, 0.5$ and 1.

5. Time-fractional Gardner's Ostrovsky Equation

In this section, we are solving the Gardner's Ostrovsky equation involving time-fractional derivative, i.e

$$(\phi_t^\alpha + c\phi_\xi + \alpha_1\phi\phi_\xi + \alpha_2\phi^2\phi_\xi + \beta\phi_{\xi\xi\xi})_\xi = \gamma\phi, \tag{5. 15}$$

subject to the initial condition

$$\phi(\xi,0) = A\operatorname{sech}^2\xi.$$

It can be written as

$$\phi_t^\alpha + c\phi_\xi + \alpha_1\phi\phi_\xi + \alpha_2\phi^2\phi_\xi + \beta\phi_{\xi\xi\xi} = \gamma\int\phi d\xi. \tag{5. 16}$$

The Laplace transform of Eq. (5. 15) can give the expression, i.e

$$\mathcal{L}\Big\{\phi_t^\alpha + c\phi_\xi + \alpha_1\phi\phi_\xi + \alpha_2\phi^2\phi_\xi + \beta\phi_{\xi\xi\xi}\Big\} = \gamma\mathcal{L}\Big\{\int\phi d\xi\Big\}.$$

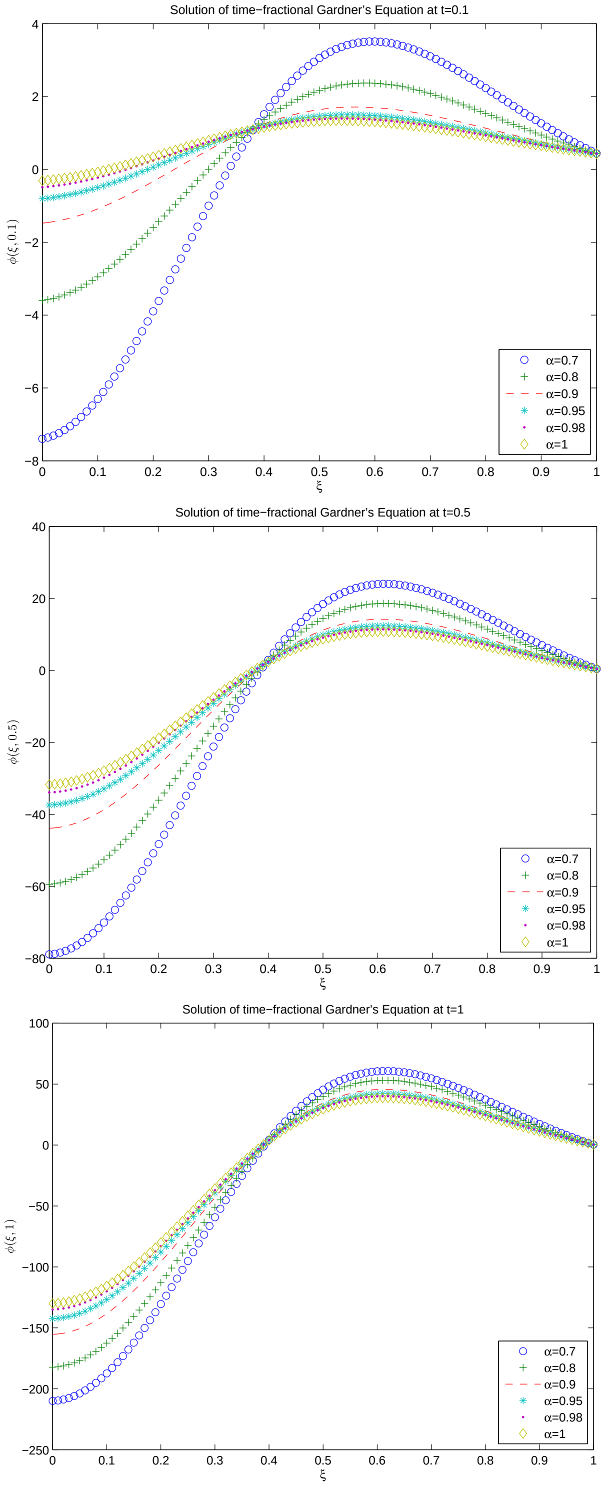
Rearranging,

$$\mathcal{L}\Big\{\phi_t^\alpha\Big\} = -c\mathcal{L}\Big\{\phi_\xi\Big\} - \alpha_1\mathcal{L}\Big\{\phi\phi_\xi\Big\} - \alpha_2\mathcal{L}\Big\{\phi^2\phi_\xi\Big\} - \beta\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\} + \gamma\mathcal{L}\Big\{\int\phi d\xi\Big\}.$$

Applying the rule of Laplace transform of fractional derivative, we get

$$s^\alpha\phi(\xi,s) - s^{\alpha-1}\phi(\xi,0) = -c\mathcal{L}\Big\{\phi_\xi\Big\} - \alpha_1\mathcal{L}\Big\{\phi\phi_\xi\Big\} - \alpha_2\mathcal{L}\Big\{\phi^2\phi_\xi\Big\} - \beta\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\} + \gamma\mathcal{L}\Big\{\int\phi d\xi\Big\}.$$

FIGURE 2. Numerical Results of Example:2 for $\alpha = 1, 0.98, 0.95, 0.9, 0.8$ and 0.7



On simplifying,

$$\phi(\xi,s) = s^{\alpha-1}\phi(\xi,0) + s^{-\alpha}\Big\{-c\mathcal{L}\Big\{\phi_\xi\Big\} - \alpha_1\mathcal{L}\Big\{\phi\phi_\xi\Big\} - \alpha_2\mathcal{L}\Big\{\phi^2\phi_\xi\Big\} - \beta\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\} + \gamma\mathcal{L}\Big\{\int\phi d\xi\Big\}\Big\}. \quad (5.\ 17)$$

On applying Inverse Laplace transform on Eq. (5. 17), we obtain

$$\mathcal{L}^{-1}\Big\{\phi(\xi,s)\Big\} = \mathcal{L}^{-1}\Big\{s^{-1}\phi(\xi,0)\Big\} + \mathcal{L}^{-1}\Big\{s^{-\alpha}\Big\{-c\mathcal{L}\Big\{\phi_\xi\Big\} - \alpha_1\mathcal{L}\Big\{\phi\phi_\xi\Big\} - \alpha_2\mathcal{L}\Big\{\phi^2\phi_\xi\Big\} - \beta\mathcal{L}\Big\{\phi_{\xi\xi\xi}\Big\} + \gamma\mathcal{L}\Big\{\int\phi d\xi\Big\}\Big\}\Big\} \quad (5.\ 18)$$

Here we put the initial condition in Eq. (5. 18), we find

$$\phi(\xi,t) = A\,\mathrm{sech}^2\,\xi + \mathcal{L}^{-1}\Big\{s^{-\alpha}\Big\{R\phi(\xi,t) + N\phi(\xi,t)\Big\}\Big\}. \quad (5.\ 19)$$

We assume that our solution can be written as power series

$$\phi(\xi,t) = \sum_{n=0}^{\infty} p^n\phi_n(\xi,t) = p^0\phi_0 + p^1\phi_1 + p^2\phi_2 + ...$$

The nonlinear term can be expressed as

$$N\phi(\xi,t) = \sum_{n=0}^{\infty} p^n H_n(\phi) = p^0H_0 + p^1H_1 + p^2H_2 + ...$$

Using the expression of He's Polynomial in nonlinearity term and the linear operator in Eq.(5. 19), we get the solution

$$\phi_0(\xi,t) = A\,\mathrm{sech}^2\,\xi,$$

$$\phi_1(\xi,t) = \frac{At^\alpha}{\Gamma(1+\alpha)}\Big\{\gamma\tanh\xi + 2(8\beta - A\alpha_1)\,\mathrm{sech}^4\,\xi\tanh\xi + 2A^2\alpha_2\,\mathrm{sech}^6\,\xi\tanh\xi - \mathrm{sech}^2\,\xi(c - 8\beta\tanh^3\xi)\Big\},$$

$$\begin{aligned}\phi_2(\xi,t) = \frac{At^{2\alpha}}{6\Gamma(1+2\alpha)}\Big\{&-12A^4\alpha_2^2\,\mathrm{sech}^{12}\,\xi + 24A^2\alpha_2\,\mathrm{sech}^{10}\,\xi(14\beta - A\alpha_1 + 5A^2\alpha_2\tanh^2\xi)\\ &- 12\,\mathrm{sech}^8\,\xi(136\beta^2 - 22A\beta\alpha_1 - 8A^2(-43\beta + 2A\alpha_1)\alpha_2\tanh^2\xi) + 6\gamma(\gamma\log(\cosh\xi)\\ &-2\alpha_1\tanh\xi + 2\beta\tanh^4\xi) - 3\,\mathrm{sech}^4\,\xi(-12\gamma\beta + 3A\alpha_1\gamma - 64\alpha_1\beta\tanh\xi + 12\alpha_1A\alpha_1\tanh\xi\\ &-8A^2\alpha_2\gamma\tanh^2\xi + 64\beta(57\beta - 5A\alpha_1)\tanh^4\xi) - 6\,\mathrm{sech}^2\,\xi(c^2 + (-4\beta\gamma + 2A\alpha_1\gamma)\tanh^2\xi\\ &- 16\alpha_1\beta\tanh^3\xi + 64\beta^2\tanh^6\xi)\Big\}.\end{aligned}$$

Adding the above approximations, that is

$$\phi(\xi,t)=\lim_{N\to\infty}\sum_{n=0}^{N}\phi_n(\xi,t)=A\operatorname{sech}^2\xi+\frac{At^{\alpha}}{\Gamma(1+\alpha)}\Big\{\gamma\tanh\xi+2(8\beta-A\alpha_1)\operatorname{sech}^4\xi\tanh\xi$$
$$+2A^2\alpha_2\operatorname{sech}^6\xi\tanh\xi-\operatorname{sech}^2\xi(c-8\beta\tanh^3\xi)\Big\}+\frac{At^{2\alpha}}{6\Gamma(1+2\alpha)}\Big\{-12A^4\alpha_2^2\operatorname{sech}^{12}\xi$$
$$+24A^2\alpha_2\operatorname{sech}^{10}\xi(14\beta-A\alpha_1+5A^2\alpha_2\tanh^2\xi)-12\operatorname{sech}^8\xi(136\beta^2-22A\beta\alpha_1$$
$$-8A^2(-43\beta+2A\alpha_1)\alpha_2\tanh^2\xi)+6\gamma(\gamma\log(\cosh\xi)-2\alpha_1\tanh\xi+2\beta\tanh^4\xi)$$
$$-3\operatorname{sech}^4\xi(-12\gamma\beta+3A\alpha_1\gamma-64\alpha_1\beta\tanh\xi+12\alpha_1A\alpha_1\tanh\xi-8A^2\alpha_2\gamma\tanh^2\xi$$
$$+64\beta(57\beta-5A\alpha_1)\tanh^4\xi)-6\operatorname{sech}^2\xi(c^2+(-4\beta\gamma+2A\alpha_1\gamma)\tanh^2\xi-16\alpha_1\beta\tanh^3\xi+64\beta^2\tanh^6\xi)\Big\}.$$

Figure:3 represents the numerical solutions of the waves obtained by solving time-fractional Gardner's Ostrovsky equation for different values of fractional order derivative. The simulation obtained from the values of α presents the nonlinear behaviour of the wave at time $t=0.1, 0.5$ and 1 to study the wave profile under the domain 0 to 1. One can easily visualize of the hidden nonlinear behaviour of the solution in Figure:1 and can be utilize for implementation.

6. Conclusion

A recent semi-analytical method HPLTM is implemented to solve Ostrovsky equation, Gardner's equation and Gardner's Ostrovsky equation involving time-fractional derivative just using the initial condition and some restrictive assumptions. The Caputo time-fractional derivative is a reliable tool to study the solution of upcoming state on the basis of all previous backgrounds of the solution. These semi-analytical results are useful for simulations of solitary wave solutions having rotational effects. This extended study of solitary theory is beneficial to understand the dynamics of solitary waves at the laboratory level.

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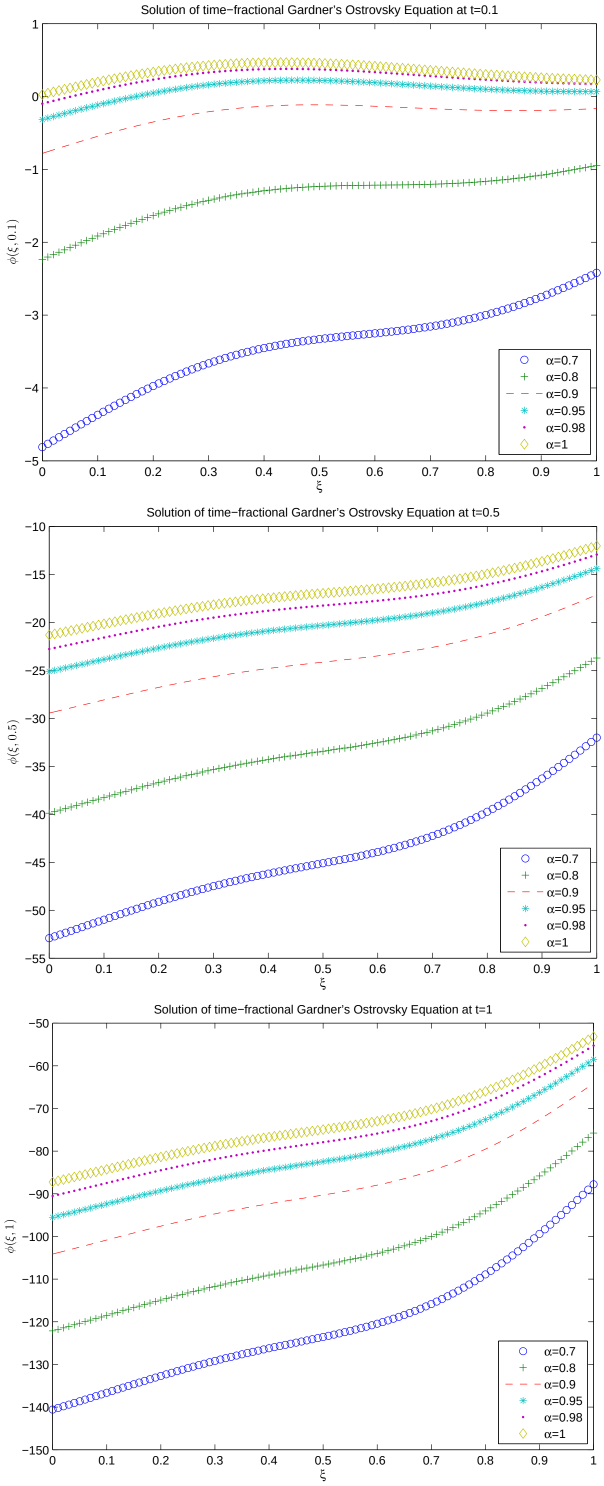
FIGURE 3. Numerical Results of Example:3 for $\alpha = 1, 0.98, 0.95, 0.9, 0.8$ and 0.7



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