Punjab University Journal of Mathematics
(ISSN 1016-2526)
Vol. 52(2)(2020) pp. 73-82

Oscillation Tests for Conformable Fractional Differential Equations with Damping

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Received: October 25, 2019 / Accepted: December 31, 2019 / Published online: 01 Febru-ary, 2020

Abstract.:This article concerns existence of oscillatory solutions of the conformable fractional equations with damping of the form

$$\left(\ell\left(y^{(\alpha)}\right)^{\gamma}\right)^{(\beta)}(s)+g\left(s,x_{\mu}^{\gamma}(s)\right)=0, \qquad \text{for all } s\in J_0,$$

where $y^{(\alpha)}$ denotes conformable fractional, $y(s)=x(s)+h(s)x_{\xi}(x)$, $x_{\mu}=x\circ\mu$, $x_{\xi}=x\circ\xi$, $\gamma:=\dfrac{2k+1}{2m+1}$, with $k,m\in\mathbb{N}$, $J_0=[0,\infty)$ and $\alpha,\beta\in(0,1]$.

AMS (MOS) Subject Classification Codes: 34K11; 39A10; 39A99; 34C10; 39A11.
Key Words: Oscillation, Dynamic equations, Riccati Technique, Fractional Calculus.

1. Introduction

Consider the conformable fractional equations with damping of the form

$$\left(\ell\left(y^{(\alpha)}\right)^{\gamma}\right)^{(\beta)}(s)+g\left(s,x_{\mu}^{\gamma}(s)\right)=0, \qquad \text{for all } s\in J_0, \tag{1. 1}$$

where $y^{(\alpha)}$ denotes conformable fractional defined in [20], $y(s):=x(s)+h(s)x_{\xi}(s)$, $x_{\mu}=x\circ\mu$, $\gamma:=\dfrac{2k+1}{2m+1}$, with $k,m\in\mathbb{N}$, $J_0=[0,\infty)$ and $\alpha,\beta\in(0,1]$. Equation (1.1) will be studied under the following assumptions:

(C_1) $g : J_0 \times \mathbb{R} \to \mathbb{R}$, such that $g \in \mathcal{C}(J_0 \times \mathbb{R}, \mathbb{R})$, $xg(s,x) > 0$, for all $(s,x) \in J_0 \times \mathbb{R}$ and there is $c \in \mathcal{C}(J_0, J_0)$, such that

$$\frac{g(s,x)}{x} \geq c(s) \qquad \text{for all } (s,x) \in J_0 \times \mathbb{R} \setminus \{0\}.$$

(C_2) $\ell, h, \xi, \mu \in \mathcal{C}(J_0, J_0)$, such as μ and ξ tends to $+\infty$, for s large enough, and

$$\xi(s) \leq s \leq \mu(s) \qquad \text{for all } s \in J_0.$$

The solution of equation (1.1) we mean a nontrivial real-valued function $y \in \mathcal{C}^{\alpha}([T_y, +\infty), \mathbb{R})$ and $\ell\left(y^{(\alpha)}\right)^{\gamma} \in \mathcal{C}^{\beta}([T_y, +\infty), \mathbb{R})$, $T_y > 0$, which satisfies (1.1) on $[T_y, +\infty)$.

The theory of the conformable fractional derivative was introduced by Khalil et al. [20] to generalize the differentiation operator in order to obtain the local fractional derivative α, such as $\alpha \notin \mathbb{N}$. Some articles are very interesting on the topics of fractional derivatives compliant see [7, 23] and the references therein.

In recent years, many research activities have been conducted on the oscillation of differential equations, including the theory of the oscillation of differential equations are applied to the study of oscillation phenomena in the fields of technology, natural sciences and social sciences. For example, in medicine (cardiac sinusoidal rhythm), electricity (free oscillations of an LC^2 circuit), physics (the theory of fluid dynamics in astrophysics) and in chemistry (oscillating reactions-chemical waves), etc. In recent years, many research activities have been conducted on the oscillation of solutions of various dynamic equations.

2. Preliminaries

In the first part of the preliminary, we present the definition and properties of the α-differentiable and α integral in the conformal sense, puls to see [20].
We denote

$$J_{s_0} := [s_0, \infty) \qquad \text{for all } s_0 \in [0, \infty).$$

Definition 2.1. [20] *Let $u : J_0 \to \mathbb{R}$ and $\alpha \in (0,1]$, We define $T_\alpha(u)(s)$ to be the number, provided it exists, such that*

$$T_\alpha(u)(s) := \lim_{\varepsilon \to 0} \frac{u\left(s + \varepsilon s^{1-\alpha}\right) - u(s)}{\varepsilon} \qquad \textit{for all } s \in J_0.$$

*Often, we write $u^{(\alpha)}$ instead of $T_\alpha(u)$ to designate the **conformable fractional derivative** of u of order α.*
In addition, if $u^{(\alpha)}$ exists, then we simply say that u is α-differentiable.
If u is α-differentiable in some $s \in (0,a)$, $a > 0$, and $\lim_{s \to 0^+} T_\alpha(u)(s)$ exists, then we define

$$u^{(\alpha)}(0) = \lim_{s \to 0^+} T_\alpha(u)(s).$$

Let $u : J_0 \to \mathbb{R}$ and $\alpha \in (0,1]$. The **conformable fractional integral** of u of order α from a to s, denoted by $I_a^{\alpha}(u)(s)$, is defined by

$$I_a^{\alpha}(u)(s) := \int_a^s \frac{u(\tau)}{\tau^{1-\alpha}} d\tau = \int_a^s u(\tau)\, d_\alpha \tau,$$

where the above integral is the usual improper Riemann integral.

Theorem 2.2. [20] *Let $\alpha \in (0,1]$ and assume u, v to be α-differentiable. Then,*

(1) $T_\alpha (au + bv) = aT_\alpha (u) + bT_\alpha (v)$ *for all* $a, b \in \mathbb{R}$,
(2) $T_\alpha (uv) = uT_\alpha (v) + T_\alpha (u) v$,
(3) $T_\alpha \left(\frac{u}{v}\right) = \frac{1}{v^2} (T_\alpha (u) v - uT_\alpha (v))$.

If, in addition, u is differentiable at a point $s > 0$, then $T_\alpha (u) = s^{1-\alpha} u^{'} (s)$.

Remark 2.3. *By Theorem 2.2 it follows that if $u \in \mathcal{C}^1 (J_a, \mathbb{R})$, then one has*

$$\lim_{\alpha \to 1} T_\alpha (u) (s) = u^{'} (s), \quad \textit{for all } s \in J_a.$$

Theorem 2.4. *If u is a continuous function in the domain of I_a^α, then*

$$T_\alpha (I_a^\alpha u (s)) = u (s), \quad \textit{for all } s \geq a.$$

3. Auxiliary result

Before stating the main results, the following definition and lemma are used.

Definition 3.1. *Let $u : J_0 \to \mathbb{R}$, we say that u is non-oscillating on J_0, If one of the conditions is true*

$i)$ $x (s) > 0$, *for s large enough.*
$ii)$ $x (s) < 0$, *for s large enough.*

Otherwise it is oscillating.

Let $\alpha \in (0, 1]$, for simplification, we note

$$\mathcal{C}^\alpha (J_0, \mathbb{R}) := \left\{ u : J_0 \to \mathbb{R} : u \text{ is } \alpha\text{-differentiable and } u^{(\alpha)} \in \mathcal{C} (J_0, \mathbb{R}) \right\}.$$

We put

$$\mathcal{E} (J_0, \mathbb{R}) := \{x : J_0 \to \mathbb{R}, \text{ such as } x (s) > 0, \text{ for } s \text{ large enough}\}.$$

Lemma 3.2. [31] *Let $u \in \mathcal{C}^\alpha (J_0, \mathbb{R})$, such that $u^{(\alpha)} (s) \geq 0$, for all $s \in J_0$, then u is increasing on J_0.*

Lemma 3.3. *If x is a solution of (1.1), such as $x \in \mathcal{E} (J_0, \mathbb{R})$. Then there are the following two cases, for $s \in J_{s_*}$, where $s_* \geq 0$ sufficiently large*

(1) $\left(\ell \left(y^{(\alpha)}\right)^\gamma\right)^{(\beta)} (s) \leq 0, y^{(\alpha)} (s) \geq 0,$
(2) $\left(\ell \left(y^{(\alpha)}\right)^\gamma\right)^{(\beta)} (s) \leq 0, y^{(\alpha)} (s) \leq 0.$

Proof. If $x \in \mathcal{E} (J_0, \mathbb{R})$, then it exists $s_* \in J_0$, such as $x_\mu (s) > 0$, for all $s \geq s_*$. From (1.1) and (C_1), we have

$$\left(\ell \left(y^{(\alpha)}\right)^\gamma\right)^{(\beta)} (s) \leq -c (s) x_\mu^\gamma (s) < 0 \qquad \text{for all } s \in J_{s_*}.$$

According to the Lemma 3.2, deduce that the function $y^{(\alpha)}$ is constant sign eventually. □

4. Oscillation results

In this section, we use the preceding hypotnosed and some sufficient conditions to find that each solution of equation (1.1) is oscillating.

Theorem 4.1. *Assume that there exist functions $\varrho, \kappa \in \mathcal{C}^{\beta}(J_0, J_0)$, such as, s_* large enough,*

$$\lim_{s\to\infty} I_{s_*}^{\beta}\Psi(s) = \infty \qquad \text{and} \qquad \lim_{s\to\infty} I_{s_*}^{\beta}\Phi(s) = \infty. \tag{4. 2}$$

where

$$\begin{aligned}
\Psi(s) &:= \varrho(s)\{1-h_\mu(s)\}^{\gamma} - \frac{s^{\gamma(\beta-\alpha)}\ell(s)\left[\varrho_+^{(\beta)}(s)\right]^{1+\gamma}}{(\gamma+1)^{\gamma+1}\varrho^{\gamma}(s)},\\
\Phi(s) &:= \kappa(s)\psi(s) + \eta(s) - [\varphi(s)]^{\gamma+1}\frac{\ell(s)\kappa(s)}{s^{(\beta-\alpha)\gamma}} - \gamma\frac{\ell^{\frac{-1}{\gamma}}(s)}{\Gamma_\alpha^{\gamma+1}(s)},\\
\psi(s) &:= c(s)\left[1-h(\mu(s))\frac{\Gamma_\alpha(\xi_\mu(s))}{\Gamma_\alpha(\mu(s))}\right]\frac{\Gamma_\alpha(\mu(s))}{\Gamma_\alpha(s)},\\
\varphi(s) &:= \frac{s^{\alpha-\beta}}{\ell^{\frac{1}{\gamma}}(s)\kappa^{\frac{1}{\gamma}}(s)\Gamma_\alpha(s)} + \frac{\kappa^{(\beta)}(s)}{(\gamma+1)\kappa(s)},\\
\eta(s) &:= \frac{s^{\alpha-\beta}}{\ell^{\frac{1}{\gamma}}(s)\kappa^{\frac{1}{\gamma}}(s)\Gamma_\alpha^{\gamma+1}(s)}, \quad \Gamma_\alpha(s) := \lim_{t\to\infty} I_s^{\alpha}(\ell^{-\frac{1}{\gamma}})(t).
\end{aligned}$$

So each solution x of equation (1.1) is oscillating.

Proof. Suppose instead that x is a solution of (1.1), such as $x \in \mathcal{E}(J_0, \mathbb{R})$.
So there is $s_* \in J_0$, such as

$$x(s) > 0, \quad x_\xi(s) > 0 \quad \text{and} \quad x_\mu(s) > 0, \quad \text{for all } s \in J_{s_*}.$$

Suppose first that y satisfies (1) of lemma 3.3.
Let

$$\Theta(s) := \varrho(s)\frac{\ell(s)\left(y^{(\alpha)}\right)^{\gamma}(s)}{y^{\gamma}(s)}, \qquad \text{for all } s \in J_{s_*}.$$

Then $\Theta \in \mathcal{C}(J_{s_*}, J_0)$. From Theorem 2.2, we have

$$\begin{aligned}
\Theta^{(\beta)}(s) &= \frac{\varrho^{(\beta)}(s)}{\varrho(s)}\Theta(s) + \varrho(s)\frac{\left(\ell\left(y^{(\alpha)}\right)^{\gamma}\right)^{(\beta)}(s)}{y^{\gamma}(s)} - \varrho(s)\frac{\ell(s)\left(y^{(\alpha)}(s)\right)^{\gamma}(y^{\gamma})^{(\beta)}(s)}{y^{2\gamma}(s)}\\
&\leq \frac{\varrho^{(\beta)}(s)}{\varrho(s)}\Theta(s) - \varrho(s)c(s)\frac{x^{\gamma}(\mu(s))}{y^{\gamma}(s)} - \gamma\varrho(s)\frac{\ell(s)\left(y^{(\alpha)}(s)\right)^{\gamma}y^{(\beta)}(s)}{y^{\gamma+1}(s)}.
\end{aligned}$$

From Theorem 2.4, we have

$$\begin{aligned} I_a^{\alpha}\left(s^{\beta-\alpha}y^{(\beta)}(s)\right) &= \int_a^s \tau^{\beta-\alpha}\frac{y^{(\beta)}(\tau)}{\tau^{1-\alpha}}d\tau \\ &= \int_a^s \frac{y^{(\beta)}(\tau)}{\tau^{1-\beta}}d\tau \\ &= I_a^{\beta}\left(y^{(\beta)}(s)\right) \\ &= y(s), \end{aligned}$$

then, we get

$$\begin{aligned} y^{(\alpha)}(s) &= \left\{I_a^{\alpha}\left(s^{\beta-\alpha}y^{(\beta)}(s)\right)\right\}^{(\alpha)} \\ &= s^{\beta-\alpha}y^{(\beta)}(s). \end{aligned} \tag{4. 3}$$

Therefore, if $y^{(a)} \in \mathcal{C}(J_{s_*}, J_0)$, by Lemma 3.2, we have y is a increasing function on J_{s_*}. As $\xi(s) \geq s$, we obtain

$$\begin{aligned} x(s) &= y(s) - h(s)\,x_{\xi}(s) \\ &\geq y(s) - h(s)\,y_{\xi}(s) \\ &\geq (1-h(s))\,y(s). \end{aligned}$$

Thus,

$$\Theta^{(\beta)}(s) \leq \frac{\varrho^{(\beta)}(s)}{\varrho(s)}\Theta(s) - \varrho(s)\left[1-h_{\mu}(s)\right]^{\gamma} - \frac{\gamma s^{\alpha-\beta}}{\ell^{\frac{1}{\gamma}}(s)\,\varrho^{\frac{1}{\gamma}}(s)}\Theta^{1+\frac{1}{\gamma}}(s).$$

Using inequality [5]

$$Ax - Bx^{1+\frac{1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}\frac{A^{1+\alpha}}{B^{\alpha}} \qquad \text{for all } x, A, B, \alpha > 0. \tag{4. 4}$$

Then

$$\Theta^{(\beta)}(s) = -\varrho(s)\left[1-h_{\mu}(s)\right]^{\gamma} + \frac{s^{\gamma(\beta-\alpha)}\ell(s)\left[\varrho_+^{(\beta)}(s)\right]^{1+\gamma}}{(\gamma+1)^{\gamma+1}\varrho^{\gamma}(s)} = -\Psi(s).$$

Then, we obtain

$$\begin{aligned} I_{s_*}^{\beta}\Psi(s) &\leq -I_{s_*}^{\beta}\Theta^{(\beta)}(s) \leq -\Theta(s) + \Theta(s_*) \\ &\leq \Theta(s_*), \end{aligned}$$

which contradicts with (4.2).

Secondly suppose that y satisfies (2) of lemma 3.3.

By (1.1), then the function $\ell\left(y^{(\alpha)}\right)^{\gamma}$ is decreasing on J_{s_*}, therefore, for any $\tau \geq s \geq s_*$, we have

$$y^{(a)}(\tau) \leq \left\{\frac{\ell(s)}{\ell(\tau)}\right\}^{\frac{1}{\gamma}} y^{(a)}(s),$$

from Theorem 2.4, we have

$$\begin{aligned} y(s) &\geq -\{\ell(s)\}^{\frac{1}{\gamma}} y^{(a)}(s) \int_s^{\infty} \left\{\frac{1}{\ell(\tau)}\right\}^{\frac{1}{\gamma}} d_\alpha \tau \\ &= -\{\ell(s)\}^{\frac{1}{\gamma}} \Gamma_\alpha(s) y^{(a)}(s), \end{aligned} \tag{4. 5}$$

by Theorem 2.2, we get

$$\Gamma_\alpha^{(a)}(s) = s^{1-\alpha}\Gamma_\alpha'(s) = -\left\{\frac{1}{\ell(s)}\right\}^{\frac{1}{\gamma}}, \tag{4. 6}$$

then

$$\left(\frac{y(s)}{\Gamma_\alpha(s)}\right)^{(\alpha)} = \frac{y^{(a)}(s)\Gamma_\alpha(s) - y(s)\Gamma_\alpha^{(\alpha)}(s)}{\Gamma_\alpha^2(s)} > 0.$$

So the function $\dfrac{y}{\Gamma_\alpha}$ is decreasing on J_{s_*}, then

$$\begin{aligned} x(s) &\geq y(s) - h(s) y_\xi(s) \\ &\geq \left[1 - h(s)\frac{\Gamma_\alpha(\xi(s))}{\Gamma_\alpha(s)}\right] y(s). \end{aligned} \tag{4. 7}$$

We havethe function y is decreasing on J_{s_*}. As $\mu(s) \leq s$, we obtain

$$\begin{aligned} x(\mu(s)) &\geq \left[1 - h(\mu(s))\frac{\Gamma_\alpha(\xi(\mu(s)))}{\Gamma_\alpha(\mu(s))}\right] y(\mu(s)) \\ &\geq \left[1 - h(\mu(s))\frac{\Gamma_\alpha(\xi(\mu(s)))}{\Gamma_\alpha(\mu(s))}\right] y(s). \end{aligned} \tag{4. 8}$$

We pose

$$\mathcal{V}(s) := \kappa(s)\left[\frac{\ell(s)(y^\alpha)^\gamma(s)}{y^\gamma(s)} + \frac{1}{\Gamma_\alpha^\gamma(s)}\right] \qquad \text{for all } s \in J_{s_*},$$

by (4.5), then $\mathcal{V}(s) > 0$, for all $s \in J_{s_*}$. Applying Theorem 2.4, we find the following relationship

$$\begin{aligned} \mathcal{V}^{(\beta)}(s) &= \frac{\kappa^{(\beta)}(s)}{\kappa(s)}\mathcal{V}(s) + \kappa(s)\left[\frac{\ell(s)(y^\alpha)^\gamma(s)}{y^\gamma(s)} + \frac{1}{\Gamma_\alpha^\gamma(s)}\right]^{(\beta)} \\ &\leq \frac{\kappa^{(\beta)}(s)}{\kappa(s)}\mathcal{V}(s) - \kappa(s)c(s)\frac{x_\mu^\gamma(s)}{y^\gamma(s)} \\ &\quad -\gamma\kappa(s)\frac{\ell(s)(y^\alpha)^\gamma(s)y^{(\beta)}(s)}{y^{\gamma+1}(s)} - \frac{(\Gamma_\alpha^\gamma(s))^{(\beta)}}{\Gamma_\alpha^{2\gamma}(s)}. \end{aligned}$$

In view (4.3) and (4.8), we find

$$\begin{aligned} \mathcal{V}^{(\beta)}(s) &\leq \frac{\kappa^{(\beta)}(s)}{\kappa(s)}\mathcal{V}(s) - \kappa(s)c(s)\frac{x_\mu^\gamma(s)}{y^\gamma(s)} \\ &\quad -\frac{\gamma s^{\alpha-\beta}}{\ell^{\frac{1}{\gamma}}(s)\kappa^{\frac{1}{\gamma}}(s)}\left[\mathcal{V}(s) - \frac{1}{\Gamma_\alpha^\gamma(s)}\right]^{1+\frac{1}{\gamma}} + \gamma\left\{\frac{1}{\ell(s)}\right\}^{\frac{1}{\gamma}}\frac{1}{\Gamma_\alpha^{\gamma+1}(s)}. \end{aligned}$$

Using inequality [25],

$$(A-B)^{1+\frac{1}{\gamma}} \geq A^{1+\frac{1}{\gamma}} - \frac{B^{\frac{1}{\gamma}}}{\gamma}\left[(\gamma+1)A-B\right] \qquad AB \geq 0.$$

If we choose

$$A_s := \mathcal{V}(s) \qquad \text{and} \qquad B_s := \frac{1}{\Gamma_\alpha^\gamma(s)}.$$

We obtain the following inequality,

$$\begin{aligned} \mathcal{V}^{(\beta)}(s) \quad &\leq \quad -\kappa(s)\,c(s)\,\frac{x_\mu^\gamma(s)}{y^\gamma(s)} + \left[\frac{(\gamma+1)\,s^{\alpha-\beta}}{\ell^{\frac{1}{\gamma}}(s)\,\kappa^{\frac{1}{\gamma}}(s)\,\Gamma_\alpha(s)} + \frac{\kappa^{(\beta)}(s)}{\kappa(s)}\right]\mathcal{V}(s) \\ &\quad -\frac{\gamma s^{\alpha-\beta}}{\ell^{\frac{1}{\gamma}}(s)\,\kappa^{\frac{1}{\gamma}}(s)}\mathcal{V}^{1+\frac{1}{\gamma}}(s) + \gamma\frac{\ell^{\frac{-1}{\gamma}}(s)}{\Gamma_\alpha^{\gamma+1}(s)} - \frac{s^{\alpha-\beta}}{\ell^{\frac{1}{\gamma}}(s)\,\kappa^{\frac{1}{\gamma}}(s)\,\Gamma_\alpha^{\gamma+1}(s)}. \end{aligned}$$

By (4.7) and (4.7), we obtain

$$\begin{aligned} \mathcal{V}^{(\beta)}(s) \quad &\leq \quad -\kappa(s)\,\psi(s) + \gamma\frac{\ell^{\frac{-1}{\gamma}}(s)}{\Gamma_\alpha^{\gamma+1}(s)} - \eta(s) \\ &\quad + \left[\frac{(\gamma+1)\,s^{\alpha-\beta}}{\ell^{\frac{1}{\gamma}}(s)\,\kappa^{\frac{1}{\gamma}}(s)\,\Gamma_\alpha(s)} + \frac{\kappa^{(\beta)}(s)}{\kappa(s)}\right]\mathcal{V}(s) - \frac{\gamma s^{\alpha-\beta}}{\ell^{\frac{1}{\gamma}}(s)\,\kappa^{\frac{1}{\gamma}}(s)}\mathcal{V}^{1+\frac{1}{\gamma}}(s). \end{aligned}$$

Using inequality (4.4), we get

$$\mathcal{V}^{(\beta)}(s) \leq -\kappa(s)\,\psi(s) - \eta(s) + \varphi^{\gamma+1}(s)\,\frac{\ell(s)\,\kappa(s)}{s^{(\alpha-\beta)\gamma}} + \gamma\frac{\ell^{\frac{-1}{\gamma}}(s)}{\Gamma_\alpha^{\gamma+1}(s)} = -\Phi(s).$$

Then, we obtain

$$I_{s_*}^\beta\,\Phi(s) \leq -I_{s_*}^\beta\,\mathcal{V}^{(\beta)}(s) \leq \mathcal{V}(s_*).$$

which contradicts with (4.2). □

Theorem 4.2. *Assume that there a function $\varrho \in \mathcal{C}^\beta(J_0, J_0)$ such as*

$$\lim_{s\to\infty} I_{s_*}^\beta\,\Psi(s) = +\infty \qquad \textit{and} \qquad \lim_{s\to\infty} I_{s_*}^\alpha\left(\ell^{-\frac{1}{\gamma}}(s)\right) = +\infty. \tag{4. 9}$$

So each solution x of equation (1.1) is oscillating.

Proof. Suppose instead that x solution to an equation (1.1), such as $x \in \mathcal{E}(J_0, \mathbb{R})$.
So there is $s_* \geq 0$, such as

$$x(s) > 0, \quad x_\xi(s) > 0 \quad \text{and} \quad x_\mu(s) > 0 \quad \text{for all } s \in J_{s_*}.$$

By Equ (1.1), we deduce that function $\ell\left(y^{(\alpha)}\right)$ is decreasing on J_{s_*}, and from it we find that function $\ell\left(y^{(\alpha)}\right)^\gamma \in \mathcal{C}(J_{s_*}, J_0)$. If not, it means there is $\overline{s} \in J_{s_*}$ such as

$$\ell(s)\left(y^{(\alpha)}\right)^\gamma(s) \leq -\sigma \qquad \text{for all } s \in J_{\overline{s}},$$

where $\sigma > 0$. Then, we obtain

$$I_{\overline{s}}^\alpha\left(y^{(\alpha)}(s)\right) \leq -\sigma I_{\overline{s}}^\alpha\left(\ell^{\frac{-1}{\gamma}}(s)\right) \qquad \text{for all } s \in J_{\overline{s}},$$

Thus the following inequality can be concluded

$$y(s) \leq y(\overline{s}) - \sigma I^{\alpha}_{\overline{s}} \left(\ell^{\frac{-1}{\gamma}}(s) \right) \qquad \text{for all } s \in J_{\overline{s}},$$

this gives $y(s)$ tends to $-\infty$, for s large enough, which gives the contradiction with $x(s) > 0$, for s large enough.
We have found or $s \in J_{s_*}$, where $s_* \geq 0$ sufficiently large

$$y^{(\alpha)}(s) \geq 0, \qquad \text{for } s \in J_{s_*}.$$

We conclude that there is one case, this case (1) of lemma 3.3. The proof is the same as that of Case (1) in Theorem 4.1, and so is omitted. This completes the proof. □

5. EXAMPLES

In the following, we illustrate possible applications with two example

Example 5.1. *Consider the differential equation following*

$$\left[\left(x(s) + \frac{1}{2} x_{\frac{s}{2}}(s) \right)^{\left(\frac{1}{2}\right)} (s) \right]^{\left(\frac{1}{4}\right)} + s^{-3} x_{2s}(s) = 0 \qquad \textit{for all } s \in J_0, \tag{5. 10}$$

here, $\alpha = \dfrac{1}{2}$, $\beta = \dfrac{1}{4}$, $\gamma = 1$, $\xi(s) = \dfrac{s}{2}$, $\mu(s) = 2s$ *and* $\ell(s) = 1$, $h(s) = \dfrac{1}{2}$.
Then $c(s) = \dfrac{1}{s^3}$ *and we have*

$$I^{\alpha}_{s_*}\left(\ell^{-\frac{1}{\gamma}}(s) \right) = I^{\frac{1}{2}}_{s_*}(1) = \int_{s_*}^{s} s^{-1/2} ds \simeq \frac{\sqrt{s}}{2}, \quad \textit{for } s \textit{ large enough.}$$

Set $\varrho(s) := s$, *we get* $\varrho^{\left(\frac{1}{4}\right)}(s) = s^{\frac{3}{4}}$ *and*

$$\Psi(s) = \frac{1}{4}\left(2s - s^{\frac{1}{4}} \right), \qquad \textit{for all } s \in J_0.$$

Then

$$I^{\beta}_{s^*} \Psi(s) \simeq \frac{s^{\frac{5}{4}}}{10}, \qquad \textit{for } s \textit{ large enough,}$$

Thus, (4.9) *hold. By Theorem 4.2, equation* (5.10) *is oscillatory.*

Example 5.2. *Consider a second-order half-linear delay dynamic equation*

$$\left[x(s) + \lambda x\left(\tfrac{s}{2}\right) \right]'' + s^2 x(2s) = 0 \qquad \textit{for all } s \in J_0, \tag{5. 11}$$

here, $\alpha = \beta = \gamma = 1$, $\xi(s) = \dfrac{s}{2}$, $\mu(s) = 2s$, $\ell(s) = 1$, *and* $h(s) = \lambda \in (0,1)$.
Then $c(s) = s^2$ *and we have*

$$I^{\alpha}_{s_*}\left(\ell^{-\frac{1}{\gamma}}(s) \right) = I^{1}_{s_*}(1) = (s - s_*) \simeq \frac{s}{2}, \quad \textit{for } s \textit{ large enough.}$$

Set $\varrho(s) := 1$, *we get*

$$\Psi(s) = 1 - \lambda, \qquad \textit{for all } s \in J_0.$$

Then

$$I^{1}_{0}\Psi(s) = I^{\beta}_{0}\Psi(s) = (1 - \lambda)s, \qquad \textit{for all } s \in J_0.$$

Thus, (4.9) *hold. By Theorem 4.2, equation* (5.11) *is oscillatory.*

6. CONCLUSION

In the manuscript, we have studied the oscillations of the solutions of the conformable fractional equations with damping, it's a generalization of the equation of the form

$$\left(\ell(y^{(1)})^{\gamma}\right)^{(1)}(s) + g\left(s, x_{\mu}^{\gamma}(s)\right) = 0 \qquad \text{for all } s \in J_0,$$

as particular case, for $\alpha = \beta = 1$.

7. ACKNOWLEDGMENTS

The work of the author is supported by the Ministry of Higher Education and Scientic Research, Algeria under number: PRFU-COOL03CU460120180001.
The author express their sincere gratitude to the editors and referee for careful reading of the original manuscript and useful comments.

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