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On Birkhoffian systems with Poisson bracket

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Abstract.: The special non-autonomous Birkhoffian equations with consistent algebraic structure are studied and it has been shown that the special form of integrable Birkhoffian vector fields are equivalent to the Hamiltonian vector fields. For a quasi Hamiltonian equation, conserved quantities are computed from some of the symmetries. Finally the Birkhoffian evolution equations are constructed.

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1. INTRODUCTION

Birkhoffian mechanics is the generalization of Hamiltonian mechanics, which can be applied to biophysics, atomic and molecular physics, space mechanics, statistical mechanics, engineering and so on [14, 15]. All conservative or non-conservative, un-constrained, non-holonomic constrained and selfadjoint systems, usually admit a representation of Birkhoff's equations [15, 7]. Recently, many results are obtained in related to the Birkhoffian mechanics such as the integral theory [8, 6], stability of motion [21, 9], inverse problem [10], as well as the symmetry [11, 20, 22, 16, 23].

Since the Birkhoffian dynamics is more general than the *Hamiltonian dynamics*, then the Birkhoffian dynamics is expected to play an important role in modern physics as they have played in non-linear systems [2, 3] and quantum systems [17, 18]. It is known that the Hamilton approximation system is used in biophysics, relativity, and the rotational relativistic system [24].

The Birkhoffian and Hamiltonian systems considered in this paper are finite dimensional. In sections 3, a special type of non-autonomous Birkhoffian systems with Poisson bracket which is called quasi-Hamiltonian system are considere and the conditions of integrability of Birkhoffian systems are obtained. In last section the Birkhoffian evolution equations constructed by such systems.

2. GENERAL PROPERTIES

The Birkhoff's equation or Birkhoffian system has the general form

$$K_{ij}(x,t)\frac{dx_j}{dt} - \left(\frac{\partial B(x,t)}{\partial x_i} + \frac{\partial R_i(x,t)}{\partial t}\right) = 0. \tag{2. 1}$$

Where $K_{ij} := \partial R_j/\partial x_i - \partial R_i/\partial x_j$. The function $B(x,t)$ is called the Birkhoffian and functions R_i, $i = 1,2,3,\cdots,2n$, are called Birkhoffian functions.

In autonomous Birkhoffian system the functions R_i and B and in semi autonomous system the functions R_i do not have explicit dependence on time variable. The Birkhoffian system (2. 1) is nonautonomous when both the functions R_i and B depend explicitly on time [15].

The Birkhoffian system is regular if $det(K_{ij})(\tilde{\Re}) \neq 0$ in the region considered. The necessary and sufficient condition for an analytic first order system

$$K_{ij}(x,t)\frac{dx_i}{dt} + D_j(x,t) = 0 \qquad i = 1,2,\cdots,2n,$$

to be self adjoint on star shaped region $\tilde{\Re}^*$ of points of $T^*M \times \mathbb{R}$, is that satisfies [18]:

$$\begin{aligned} &\frac{\partial K_{ij}}{\partial x_k} + \frac{\partial K_{jk}}{\partial x_i} + \frac{\partial K_{ki}}{\partial x_j} = 0, && K_{ij} + K_{ji} = 0, \\ &\frac{\partial Kij}{\partial t} = \frac{\partial D_j}{\partial x_i} - \frac{\partial D_i}{\partial x_j}, && i,j,k = 1,2,\cdots,2n. \end{aligned} \tag{2. 2}$$

Here, T^*M is the cotengent bundle of the $2n-$dimentional manifold M. Let $D_j(x,t) = \partial B/\partial x_j + \partial R_j/\partial t$, clearly the Birkhoff system (2. 1) will be self adjoint. For autonomous and semi autonomous Birkhoffian systems, consider a symplectic two form in local coordinates

$$\Omega = \sum_{i,j=1}^{2n} K_{ij}(x,t)dx_i \wedge dx_j. \tag{2. 3}$$

Note that the form (2. 3) has the property $d\Omega = 0$.

For non-autonomous case, all of conditions in (2. 2) must be considered and the coresponding geometric structure can be obtained by replacing symplectic geometry on T^*M with local coordinates x_i by contact geometry on the manifold $M \times \mathbb{R}$ with local coordinates $\tilde{x}_i$, $i = 1,\cdots,2n+1$,$\tilde{x}_{2n+1} = t$. Then we have an exact contact $2-$form

$$\hat{\Omega} = \sum_{i,j=1}^{2n+1} \hat{K}_{ij} d\tilde{x}_i \wedge d\tilde{x}_j = \Omega + 2D_i dx_i \wedge dt, \qquad \hat{\mathbf{K}} = \begin{pmatrix} K & D \\ -D^T & 0 \end{pmatrix}, \tag{2. 4}$$

in (2n+1) dimentional space. On the manifold $T^*M \times \mathbb{R}$, we have the contact form $\eta = \tilde{R}_i(\tilde{x})d\tilde{x}_i$ such that

$$\tilde{\mathbf{R}}_\mathbf{i} = \begin{cases} -B & i = 2n+1 \\ R_i & i \neq 2n+1 \end{cases}$$

It can be seen $d\eta = \hat{\Omega}$. Moreover, the non-autonomous system

$$\frac{d\tilde{x}_i}{dt} = -B(x,t)/\tilde{R}_i(x,t),$$

is the geodesic equation on manifold $M \times \mathbb{R}$.

3. Algebraic properties

For non-autonomous Birkhoff's equation (2. 1) we define an algebraic product

$$\frac{\partial A}{\partial x_i} K^{ij} \Big(\frac{\partial B}{\partial x_j} + \frac{\partial R_j}{\partial t} \Big) := A.B, \tag{3. 5}$$

where K^{ij} is the covariant tensor of K_{ij} and A, B are Birkhoffian.

In autonomous and semi-autonomous Birkhoff systems product (3. 5), satisfies the left and right distributive and scalar laws

$$\begin{aligned} &A.(B+C) = A.B + A.C, && (A+B).C = A.C + BC, \\ &(\lambda A).B = A.(\lambda B) = \lambda (A.B) && \lambda \ \ is\ constant \end{aligned}$$

and lie algebraic axiom, so it defines a generalized Poisson bracket. We say in this case the Birkhoffian equations possess a lie algebraic structure [12, 4]. Let $x = (x_1, x_2, \cdots, x_{2n})$ be the local coordinates on M, the associated Birkhoffian vector field will be of the form $\hat{v}_B = K^{ij}(\partial B/\partial x_i)(\partial/\partial x_j)$. Then such as Hamiltonian systems, we will have $[\hat{v}_B, \hat{v}_A] = \hat{v}_{B.A}$.

But in non-autonomous case, the product (3. 5) doesn't satisfy the left distributive law, so an algebraic structure for these equations can't be expressed by the product (3. 5).

We consider the special case of non-autonomous Birkhoff's equation(2. 1) satisfying

$$K^{ij} \frac{\partial R_j}{\partial t} = T^{ij} \frac{\partial B}{\partial x_j}, \tag{3. 6}$$

where $(\mathbf{T^{ij}})$ is a diagonal matrix.

Then the Birkhoffian system (2. 1) takes the form

$$\dot{x}_i - S^{ij} \frac{\partial B}{\partial x_j} = 0, \qquad \text{where} \qquad S^{ij} = K^{ij} + T^{ij}. \tag{3. 7}$$

We define a product

$$\frac{\partial A}{\partial x_i} S^{ij} \frac{\partial B}{\partial x_j} := A * B. \tag{3. 8}$$

From the product (3. 8) we define a Poisson bracket $A \circ B = A * B - B * A$ as [4]. The special non-autonomous Birkhoff equations (3. 7) not only possess a consistent algebraic structure according to the product (3. 8) but possess a cosistent lie algebraic structure [4].

The system (3. 7) can be written as $x_t = \hat{v}_B(x) = B * x$. Let

$$(\mathbf{S^{ij}}) = \begin{pmatrix} T^{11} & \dots & 0 & -f_1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & T^{nn} & 0 & \dots & -f_n \\ f_1 & & 0 & T^{(n+1)(n+1)} & & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & f_n & 0 & \dots & T^{2n2n} \end{pmatrix},$$

where f_is are real functions on M, then we call the corresponding system a quasi Hamiltonian system.

Here the associated Birkhoffian vector field will be of the form

$$\hat{v}_B = S^{ij}\frac{\partial B}{\partial x_i}\frac{\partial}{\partial x_j}. \tag{3. 9}$$

We denote the space of Birkhoffian vector fields (3. 9) by $\mathcal{B}$. If T^{ii}s $(i = 1, \cdots, 2n)$ dont depend on X, then

$$\begin{aligned}[\hat{v}_A, \hat{v}_B] = f_j\frac{\partial A}{\partial x_j}\hat{v}_{\partial B/\partial x_{j+n}} + g_j\frac{\partial B}{\partial x_{j+n}}\hat{v}_{\partial A/\partial x_j} - f_j\frac{\partial A}{\partial x_{j+n}}\hat{v}_{\partial B/\partial x_j} \\ - g_j\frac{\partial B}{\partial x_j}\hat{v}_{\partial A/\partial x_{j+n}} + T^j\frac{\partial A}{\partial x_j}\hat{v}_{\partial B/\partial x_j} + T^{j+n}\frac{\partial A}{\partial x_{j+n}}\hat{v}_{\partial B/\partial x_{j+n}} \\ - S^j\frac{\partial B}{\partial x_j}\hat{v}_{\partial A/\partial x_j} - S^{j+n}\frac{\partial B}{\partial x_{j+n}}\hat{v}_{\partial A/\partial x_{j+n}},\end{aligned}$$

for every $\hat{v}_A, \hat{v}_B \in \mathcal{B}$. Note that, $f_j, T^j, T^{j+n}(j = 1, ..., n)$ are the entries of the matrix of $\hat{v}_A$ and g_j, S^j, S^{j+n} belong to $\hat{v}_B$.

4. Geometric properties

Since the Birkhoffian system is self adjoint, then the phase flow of the Birkhoff's equation conserves the symplecticity and we get

$$\frac{d}{dt}\Omega = \frac{d}{dt}(K_{ij}dx_i \wedge dx_j) = 0.$$

Let $(\hat{x}, \hat{t})$, be the phase flow of the Birkhoffian system (2. 1), then $K_{ij}(\hat{x}, \hat{t})d\hat{x}_i \wedge d\hat{x}_j = K_{ij}(x, t)dx_i \wedge dx_j$. Also, the symplectic form Ω is an integral invariant of any Birkhoff's equation $L_{\hat{X}_B}(\Omega) = 0$. Therefore, the Birkhoffian vector field $\hat{v}_B$ preserves the rank on manifold $M \times \mathbb{R}$ i.e the rank of $M \times \mathbb{R}$ at $(\exp(t\hat{v}_B)x, t)$ is the same as the rank of $M \times \mathbb{R}$ at (x, t).

If we consider to the canonical form of the Poisson bracket, then the Birkhoffian system (3. 7) takes the form

$$\dot{x}_i = J^{ij}\frac{\partial B}{\partial x_j} \tag{4. 10}$$

with canonical matrix

$$(\mathbf{J^{ij}}) = \begin{pmatrix} T^1 & \dots & 0 & -1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & T^n & 0 & \dots & -1 \\ 1 & & 0 & T^{n+1} & & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & T^{2n} \end{pmatrix},$$

and Birkhoffian B s.t $T^i \partial^2 B/\partial {x_i}^2 = 0$, for $i = 1, ..., 2n$.

Suppose that $\tilde{\mathcal{B}}$ is the space of Birkhoffian vector fields $\hat{v}_B = J^{ij}(\partial B/\partial x_i)\partial/\partial x_j$, then it can be easily proved that $[\hat{v}_A, \hat{v}_B] = \hat{v}_{A\circ B}$ for every $\hat{v}_A, \hat{v}_B \in \tilde{\mathcal{B}}$.

Theorem 1. *The Birkhoffian flow of special type non-autonomous Birkhoff equations* (4. 10) *preserves the Poisson bracket on $M \times \mathbb{R}$.*

Proof: If F, B and P are real valued functions, and $\phi_t = \exp(t\hat{v}_B)$. By differentiating the Poisson condition $F(\phi_t) \circ P(\phi_t) = F \circ P(\phi_t)$ with respect to t and using the fact

$$\frac{\partial(F * B)}{\partial x_i} = \frac{\partial F}{\partial x_i} * B + F * \frac{\partial B}{\partial x_i},$$

we find

$$\hat{v}_B(F \circ P) - \hat{v}_{F\circ P}(B) = \hat{v}_B(F) \circ P - \hat{v}_F(B) \circ P + F \circ \hat{v}_B(P) - F \circ \hat{v}_P(B),$$

at the point $\phi_t(x)$. This is the same as the Jacobi identity. □

Theorem 2. *Let $M \times \mathbb{R}$ be a Poisson manifold. The system of Birkhoffian vector fields $\tilde{\mathcal{B}}$ on $M \times \mathbb{R}$ is integrable, so through each point $(x, t) \in M \times \mathbb{R}$ there passes an integral submanifold N of $\tilde{\mathcal{B}}$ satifying $TN|_p = \tilde{\mathcal{B}}|_p$ for every $p \in N$, such that is a contact submanifold of the Poisson manifold, and, collectively, these submanifolds determine the contact foliation of $M \times \mathbb{R}$. Also, for Birkhoffian function $B : M \times \mathbb{R} \to \mathbb{R}$, any solution to the corresponding quasi Hamiltonian system, with initial data in N, remains in a single integral submanifold N.*

Proof: As mentioned, the Lie bracket of two Birkhoffian vector fields (3. 9) in canonical form is a combination of such Birkhoffian vector fields, so $\tilde{\mathcal{B}}$ is involutive. On the other hand, for non-autonomous Birkhoff equations, a contact form can be found in each point. Finally by the variable rank Frobenius' theorem and last theorem, the proof will be completed. □

Recall the similar foliation of manifold M related to Hamiltonian systems. Cosider the Hamiltonian system in which its Hamiltonian is the same as the Birkhoffian of the Birkhoffian system (4. 10). we will have these conclusions.

Theorem 3. *The phase flow of a Hamiltonian system is symplectomorphic to the phase flow of the corresponding quasi Hamiltonin system.*

Proof: Suppose that $\hat{x} = g(x, t)$ is the flow and N is the phase flow of the Birkhoffian system, and the graph of the phase flow of the Birkhoffian system is $g^t(x, t_0) = g(x, t, t_0)$, i.e.

$$\Gamma_g = \{(\hat{x}, x) \in \mathbb{R}^{4n} | \hat{x} = g(x, t, t_0), x \in \mathbb{R}^{2n}\},$$

and suppose that $\tilde{N}$ and $\tilde{g}^t(x,t_0) = \tilde{g}(x,t,t_0), \Gamma_{\tilde{g}}$ belong to the Hamiltonian system. Note that the manifold $N \times \mathbb{R}^{2n}$ is symplectic with symplectic two form

$$\tilde{\Omega} = K(\hat{x},t)d\hat{x}_i \wedge d\hat{x}_j - K(x,t_0)dx_i \wedge dx_j,$$

and we have the inclusion map $i : \Gamma_g \subset N \times \mathbb{R}^{2n}$ such that $i^*(\tilde{\Omega}) = 0$ [18]. Thus Γ_g will be a lagrangian submanifold of $N \times \mathbb{R}^{2n}$. Then $g^t : \mathbb{R}^{2n} \to N$ is a symplectomorphism[1]. Similarly we have the symplectomorphism $\tilde{g}^t : \mathbb{R}^{2n} \to \tilde{N}$ in Hamiltonian case. So $\tilde{g}^t \circ g^{-t} : N \to \tilde{N}$ is symplectomorphism. □

Now, consider a contact foliation of $M \times \mathbb{R}$ constructed from the symplectic Hamiltonian foliation of manifold M. Therefore we can say the solutions of the Birkhoffian system (4. 10) lie on hypersurfaces of contact leaves of manifold $M \times \mathbb{R}$ up to diffeomorphism.
Theorem 4. *The solution of quasi Hamiltonian system* (4. 10)*, passing through* $p \in M \times \mathbb{R}$ *lie on the hypersurface of a contact submanifold of* $M \times \mathbb{R}$ *in* p. □
Example 1. The system of equations $\dot{x}_1 = t^2 e^{x_1} \sin tx_2 + te^{x_1} \cos tx_2$ and $\dot{x}_2 = -e^{x_1} \sin tx_2 + te^{x_1} \cos tx_2$, is quasi hamiltonian whit Birkhoffian $B = e^{x_1} \sin tx_2$ and $T^{11} = t^2, T^{22} = 1$. On the other hand it is a Hamiltonian equation with Hamiltonian $H = -te^{x_1} \cos tx_2 + e^{x_1} \sin tx_2$.

5. Symmetries and conservation law

Suppose that Δ is a Birkhoff system and for$D_j = \partial B/\partial x_j + \partial R_j/\partial t$, $D^{(i)}_{\Delta} = K_{ij} D_t - D_j$, is the Frechet derivative of Δ. By conditions in (2. 2), $D^{(i)}_{\Delta}$ will be selfadjoint i.e. $D^{(i)}_{\Delta} = D^{(i)*}_{\Delta}$.

In fact, a general non-autonomous first order system is self sdjoint in region $\tilde{\Re}^*$ of points of $T\mathbb{R}^{2n} \times \mathbb{R}$ if and only if it is of the Birkhoffian type [15]. As a result of self adjointness we have the integrability condition for the contact structure $\hat{\Omega} = d(\tilde{R}_i d\tilde{x}_i)$.

The adjointness, implies that Δ is the Euler Lagrange expression for some variational problem $\mathcal{L} = \int L\, dx$. In which case $\Delta = E(L)$, a lagrangian for Δ can be constructed by integration [13], or calculated by Pfaff action [5]. Indeed for Birkhoffian system, we have $L\, dt = \eta$.

Therefore, by Noether's theorem we can get the conservation law for Birkhoffian equation from the variational symmetries of the variational problem. Also, the Noetherian symmetries form a closed subalgebra of Lie symmetries algebra [19].
Example 2. Consider the equations

$$\ddot{u}_1 = u_2, \qquad \ddot{u}_2 = u_1. \tag{5. 11}$$

Let $x_1 = u_1$, $x_2 = u_2$, $x_3 = \dot{u}_2$, $x_4 = \dot{u}_1$, then the equation (5. 11) can be transformed into the form of a quasi Hamiltonian equation with

$$R_1 = -x_3 - x_1 t, \qquad R_2 = -x_4 - x_2 t, \qquad R_3 = R_4 = 0, \qquad B = x_3 x_4.$$

The generators of symmetry algebra of the equation are $v_1 = \partial/\partial t$ and

$$v_2 = \sinh t\frac{\partial}{\partial u_1} + \sinh t\frac{\partial}{\partial u_2}, \qquad v_3 = \cosh t\frac{\partial}{\partial u_1} + \cosh t\frac{\partial}{\partial u_2},$$
$$v_4 = \sin t\frac{\partial}{\partial u_1} - \sin t\frac{\partial}{\partial u_2}, \qquad v_5 = \cos t\frac{\partial}{\partial u_1} - \cos t\frac{\partial}{\partial u_2},$$
$$v_6 = u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}, \qquad v_7 = u_2\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2}.$$

The Noetherian symmetries correspond to the spanned subalgebra by the vector fields $v_1, \cdots, v_5$. So the corresponding conservation laws for the equation (5. 11) are $p_1 = \dot{u}_1^2 + \dot{u}_2^2 - u_1u_2$, and

$$p_2 = \sinh t(\dot{u}_1 + \dot{u}_2) - \cosh t(u_1 + u_2), \quad p_3 = \cosh t(\dot{u}_1 + \dot{u}_2) - \sinh t(u_1 + u_2),$$
$$p_4 = \sin t(\dot{u}_1 - \dot{u}_2) - \cos t(u_1 - u_2), \qquad p_5 = \cos t(\dot{u}_1 - \dot{u}_2) - \sin t(u_1 - u_2).$$

6. Evolution equations

Suppose that M is an open subset of the space $X \times U$ in which $x = (x_1, \cdots, x_n) \in X$ is independent and $u = (u_1, \cdots, u_n) \in U$ is dependent variable. Suppose that $\mathcal{A}$ is the algebra of differential functions over M and $\mathcal{F}$ is the quotient space of $\mathcal{A}$ under the total divergence.

In order to construct an evolution equation of Birkhoffian type we need various components of Birkhoffian system (3. 7). For this we replace Birkhoffian function B with Birkhoffian functional $\mathscr{B} = \int B dx \in \mathcal{F}$ and gradient operation by the functional gradient $\delta\mathscr{B} \in \mathcal{A}^n$.

According to quasi Hamiltonian matrix, we suggest associated linear operator $\mathscr{D}$: $\mathcal{A}^n \to \mathcal{A}^n$ in the form $\mathscr{D} = f(t)\mathscr{D}_1 + \mathscr{D}_2$ such that $\mathscr{D}$ can be a linear $n \times n$ matrix differential operator and $f(t)$ a diagonal $n \times n$ matrix with real function entries. $\mathscr{D}_1, \mathscr{D}_2$ are self adjoint and skew adjoint(Hamiltonian) operators, respectively.

We define the corresponding poisson bracket as

$$\mathscr{R} \circ \mathscr{L} = \int \delta\mathscr{R}.(\mathscr{D} - \mathscr{D}^*)\delta\mathscr{L}\, dx. \tag{6. 12}$$

Clearly the poisson bracket satisfies of the skew symmetry $\mathscr{R} \circ \mathscr{L} = -\mathscr{L} \circ \mathscr{R}$, and the Jacobi identity $(\mathscr{P} \circ \mathscr{L}) \circ \mathscr{R} + (\mathscr{R} \circ \mathscr{P}) \circ \mathscr{L} = (\mathscr{L} \circ \mathscr{R}) \circ \mathscr{P}$ for all functionals $\mathscr{P}$, $\mathscr{L}$, and $\mathscr{R}$.

Theorem 5. *Let $\mathscr{D}$ be a Birkhoffian operator with poisson bracket* (5. 11)*. There is an evolutionary vector field $\hat{v}_{\mathscr{B}}$ which corresponds to each functional $\mathscr{B} = \int B\, dx$ and satisfies* $\mathrm{pr}\hat{v}_{\mathscr{B}}(\mathscr{R}) - \mathrm{pr}\hat{v}_{\mathscr{R}}(\mathscr{B}) = \mathscr{R} \circ \mathscr{B}$ *for all functional $\mathscr{R} \in \mathcal{F}$ (See P.J. Olver* [13]*).* □

In fact $\hat{v}_{\mathscr{B}}$ has characteristic $\mathscr{D}\delta\mathscr{B} = \mathscr{D}E(B)$. By exponentiating the Birkhoffian vector field $\hat{v}_{\mathscr{B}}$, we obtain the evolution Birkhoffian system corresponding to functional $\mathscr{B}[u]$ in the following form

$$\frac{\partial u}{\partial t} = \mathscr{D}.\delta\mathscr{B}$$

Suppose that the Birkhoffian operator $\mathscr{D}$ of the Birkhoffian evolution equation has the property $\delta\mathscr{D}_1.\delta\mathscr{B} = 0$, then for the generalized symmetry group $\hat{v}_{\mathscr{P}}$ of the system we have $[\hat{v}_{\mathscr{B}}, \hat{v}_{\mathscr{P}}] = \hat{v}_{\mathscr{B}\circ\mathscr{P}}$. For example Burger's equation $u_t = 2uu_x + u_{xx}$ is a Birkhoffian evolution equation with the operator $\mathscr{D} = 2u + D_x$ and is a Hamiltonian evolution equation too. Specially in the case of $f(t) = c$(c is constant), the functional $\mathscr{P}$ satisfies

$$\frac{\partial \mathscr{P}}{\partial t} + \mathscr{B} \circ \mathscr{P} = \mathscr{C}$$

for some distinguished functional $\mathscr{C}$ ($\mathscr{D}\delta\mathscr{C} = 0$).

So we can say $v_{\tilde{\mathscr{P}}}$ with the characteristic $(\mathscr{D} - \mathscr{D}^*)\delta\tilde{\mathscr{P}}$ determines a generalized symmetry group of the corresponding Hamiltonian evolution equation if and only if there is a functional $\tilde{\mathscr{C}}$ with the property $\mathscr{D}\delta\mathscr{C} = (\mathscr{D} - \mathscr{D}^*)\delta\tilde{\mathscr{C}}$ such that $\tilde{\mathscr{P}} = \mathscr{P} - \tilde{\mathscr{C}}$.

For example the symmetries of Burger's equation $u_t = 2uu_x + u_{xx}$ can be derived from the symmetries of the corresponding Hamiltonian evolution equation, i.e the heat equation $u_t = u_{xx}$ (See P.J. Olver [13]).

7. Conclusion

In this paper we begin the analysis of the integrable nonautonomous Birkhoffian systems and we find the conditions that these systems are equivalent to the Hamiltonian's. The study of integral manifolds of these systems provide an effective way for the study of dynamics of such systems .The result of this paper develops the evolution equations of Hamiltonian type to the Birkhoffian evolution equations and make a relation between symmetries of an evolution equation and the symmetries of its Hamiltonian evolution equation part.

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